Linear Algebra

Homework 3 of March 13, 2014

Spring Semester 2014
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Problem 1 (10%) *Right Inverse of a Non-Square Matrix*

In class, we have illustrated an example of finding the right inverse of a non-square matrix $\mathsf{A}_{2\times 3} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$ (See slides 2-52 to 2-54). Define the 2 by 2 identity matrix as $\mathsf{I}_{2\times 2} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}$. Then the right inverse $\mathsf{R}_{3\times 2}$ of $\mathsf{A}_{2\times 3}$ satisies that $\mathsf{A}_{2\times 3}\mathsf{R}_{3\times 2} = \mathsf{I}_{2\times 2}$. Assume that we have found the elimination matrix $\mathsf{E}_{2\times 2}$ by Gauss-Jordan method such that

$$\mathsf{E}_{2\times 2}\begin{bmatrix}\mathsf{A}_{2\times 3} & \mathsf{I}_{2\times 2}\end{bmatrix} = \begin{bmatrix}\mathsf{E}_{2\times 2}\mathsf{A}_{2\times 3} & \mathsf{E}_{2\times 2}\mathsf{I}\end{bmatrix} = \begin{bmatrix}\mathsf{I}_{2\times 2} & \mathsf{E}_{2\times 2}\mathbf{a}_3 & \mathsf{E}_{2\times 2}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathsf{E}_{2\times 2}\mathbf{a}_3 & \mathsf{E}_{2\times 2}\end{bmatrix}.$$

Using the above representation, show that

$$\mathsf{R}_{3\times 2} = \begin{bmatrix}\mathbf{r}'_1 \\ \mathbf{r}'_2 \\ \mathbf{r}'_3\end{bmatrix} = \begin{bmatrix}\mathsf{E}_{2\times 2} - \mathsf{E}\mathbf{a}_3\mathbf{r}'_3 \\ \mathbf{r}'_3\end{bmatrix},$$

which depends on how we choose the entries of $\mathbf{r}'_3$.

Problem 2 (10%) *Vandermonde Matrix*

Recall that the *Gauss–Jordan method* can also be used to determine the *determinat*, which is equal to the product of all the pivots before normalization (Slide 2-58). Show that the determinant of the matrix

$$\mathsf{A} \triangleq \begin{bmatrix}1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2\end{bmatrix}$$

is equal to $(b-a)(c-b)(c-a)$ with distinct values $a, b, c \in \mathbb{R}$.

Problem 3 (10%) *Singular Symmetric Matrix*

Let k be a positive integer and let I be the $k \times k$ identity matrix. Show that the symmetric matrix

$$\mathsf{A}_{k\times k} \triangleq k\mathsf{I}_{k\times k} - \begin{bmatrix}1 & 1 & \cdots & 1 \\ 1 & \cdots & & \\ \vdots & & \ddots & \\ 1 & 1 & \cdots & 1\end{bmatrix}_{k\times k}$$

is *not invertible.*

Hint: Find a nonzero vector $\mathbf{x}_{k\times 1}$ *such that* $\mathsf{A}\mathbf{x} = \mathbf{0}$.

Problem 4 (10%) LU ***of a Symmetric Matrix***

Compute the LU-factorization of the following symmetric matrix:

$$\mathsf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find conditions on a, b, c, d such that A has four pivots.

Problem 5 (30%) ***Some*** 2×2 ***Symmetric Matrices***

Find 2×2 symmetric matrices $\mathsf{A} = \mathsf{A}^\intercal$ with these properties:

a) A is not invertible.

b) A is invertible, but cannot be factored into LU (*i.e.*, row exchanges/permutation is necessary).

c) A can be factored into $\mathsf{LDL}^\intercal$, but not into $\mathsf{LL}^\intercal$. (I.e., $\mathsf{A} = \mathsf{LDL}^\intercal$ but $\mathsf{A} \neq \mathsf{LL}^\intercal$.)

Problem 6 (20%) $\mathsf{PA} = \mathsf{LU}$ ***Factorization***

Find the $\mathsf{PA} = \mathsf{LU}$ factorization for

$$\text{a)} \quad \mathsf{A}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \qquad \text{b)} \quad \mathsf{A}_2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Problem 7 (10%) ***Inverse of Permutation-like Matrix***

For a permutation matrix

$$\mathsf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

explain why $\mathsf{P}^3 = \mathsf{I}_{3\times 3}$, where I is the 3×3 identity matrix.
Next, consider

$$\mathsf{A} = \begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with $a \neq 0$. Verify that $\mathsf{A}^3 = a\,\mathsf{I}$, and find the inverse of A.