LINEAR ALGEBRA

Spring Semester 2014
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Homework 7 of April 24, 2014

Deadline: May 01, 2014

Problem 1 (30%) ***True or False***

True or false? Explain if true, or find a counterexample if false:

a) The determinant of $\mathsf{I}+\mathsf{A}$ is $1+\det(\mathsf{A})$.

b) The determinant of ABC is $|\mathsf{A}||\mathsf{B}||\mathsf{C}|$.

c) The determinant of $4\mathsf{A}$ is $4|\mathsf{A}|$.

d) The determinant of $\mathsf{AB}-\mathsf{BA}$ is zero.

   Hint: Try an example with $\mathsf{A}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$.

e) If A is not invertible, then AB is not invertible.

f) The determinant of A is always the product of its pivots.

g) AB and BA have the same determinant.

h) If E is an elementary matrix, then $\det(\mathsf{E})=\pm 1$.

i) The determinant of a skew-symmetric matrix ($\mathsf{A}^{\mathsf{T}}=-\mathsf{A}$) is 0.

j) If B is a matrix obtained by interchanging two rows or two columns of A, then $\det(\mathsf{A})=\det(\mathsf{B})$.

Problem 2 (10%) ***Reverse Identity Matrices***

The reverse identity matrix J_n is the $n\times n$ matrix

$$\mathsf{J}_n \triangleq \begin{bmatrix}0 & 0 & \cdots & 0 & 1\\ 0 & 0 & \cdots & 1 & 0\\ \vdots & \vdots & \iddots & \vdots & \vdots\\ 0 & 1 & \cdots & 0 & 0\\ 1 & 0 & \cdots & 0 & 0\end{bmatrix}.$$

E.g.,

$$\mathsf{J}_3=\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix} \Longrightarrow \det(\mathsf{J}_3)=-1$$

$$\mathsf{J}_5=\begin{bmatrix}0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0\end{bmatrix} \Longrightarrow \det(\mathsf{J}_5)=1.$$

Find $\det(\mathsf{J}_{301})$ and compute the determinant $\det(\mathsf{J}_n)$ for any positive intger n.

Problem 3 (20%) 4 ***by*** 4 ***Vandermonde Matrix***

We already know that the determinant of a 3 by 3 Vandermonde matrix

$$\mathsf{V}_3 \triangleq \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

is equal to $(b-a)(c-b)(c-a)$ with distinct values $a, b, c \in \mathbb{R}$. Consider the 4 by 4 Vandermonde matrix

$$\mathsf{V}_4(x) \triangleq \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & x \\ a^2 & b^2 & c^2 & x^2 \\ a^3 & b^3 & c^3 & x^3 \end{bmatrix},$$

where we treat x as an unknown, and a, b, c as constants. Find the determinant of $\mathsf{V}_4(x)$ by the following steps:

a) (10%) Show that the determinant of $\mathsf{V}_4(x)$ can be written as $\det(\mathsf{V}_4(x)) = K(x-a)(x-b)(x-c)$ for some constant K.

b) (10%) Determine the constant K by the Co-factor formula.

Problem 4 (20%) ***Fibonacci-Like***

Consider the following sequence of matrices:

$$\mathsf{S}_1 \triangleq \begin{bmatrix} 3 \end{bmatrix}, \qquad \mathsf{S}_2 \triangleq \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \qquad \mathsf{S}_3 \triangleq \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \qquad \mathsf{S}_4 \triangleq \begin{bmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \qquad \cdots$$

a) (5%) Compute $\det(\mathsf{S}_1)$, $\det(\mathsf{S}_2)$, $\det(\mathsf{S}_3)$, and $\det(\mathsf{S}_4)$.

b) (10%) Find a recursive formula for the determinant $\det(\mathsf{S}_n)$.

   Hint: Use the Co-factor formula on the first column.

c) (5%) Prove that the above recursion produces every second Fibonacci number, i.e., prove that $\det(\mathsf{S}_n) = \mathsf{F}_{2n+2}$, where $\mathsf{F}_0 = 0$, $\mathsf{F}_1 = 1$, and $\mathsf{F}_k = \mathsf{F}_{k-1} + \mathsf{F}_{k-2}$.

Problem 5 (20%) ***Polynomial and Determinant of Matrix***

Let a 4 by 4 matrix A have the form

$$\mathsf{A} \triangleq \begin{bmatrix} 0 & 0 & 0 & a_0 \\ -1 & 0 & 0 & a_1 \\ 0 & -1 & 0 & a_2 \\ 0 & 0 & -1 & a_3 \end{bmatrix}$$

Using the steps below to find the determinant of $\det(\mathsf{A} + x\mathsf{I})$, where I is the 4 by 4 identity matrix.

a) (10%) Using the co-factor formula to expand $\det(\mathsf{A} + x\mathsf{I})$ by the first row.

b) (10%) Show that $\det(\mathsf{A} + x\mathsf{I}) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.