Linear Algebra

Homework 8 of May 1, 2014

Spring Semester 2014
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http://shannon.cm.nctu.edu.tw/la.htm

Deadline: May 8, 2014

Problem 1 (30%) *Triple Product*

Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$.

a) Prove that

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \det\left(\begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right).$$

b) Compute the volume of the box formed by four vectors $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

Problem 2 (20%) *Proof of Cramer's Rule*

Let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the solution of $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$. Prove that

$$x_1 = \frac{\texttt{det}\left(\begin{bmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{bmatrix}\right)}{\texttt{det}\left(\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix}\right)}.$$

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Problem 3 (30%) *Cofactor Matrix*

a) Find the cofactor matrix C for

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

b) Compute $\mathsf{A}\mathsf{C}^\mathsf{T}$.

c) Change $a_{1,3}$ to 40, i.e.,

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 40 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

Then find $\det(\mathsf{A})$.

Please submit your homework either to ED 823 or to the TAs during the Thursday class time

Problem 4 (20%) ***Jacobian Matrix***

Let $x = \rho \sin(\phi) \cos(\theta)$, $y = \rho \sin(\phi) \sin(\theta)$, and $z = \rho \cos(\phi)$. Show that the determinant of

$$\begin{bmatrix} \partial x/\partial \rho & \partial x/\partial \phi & \partial x/\partial \theta \\ \partial y/\partial \rho & \partial y/\partial \phi & \partial y/\partial \theta \\ \partial z/\partial \rho & \partial z/\partial \phi & \partial z/\partial \theta \end{bmatrix}$$

is equal to $\rho^2 \sin(\phi)$.