Linear Algebra

Homework 9 of May 15, 2014

Deadline: May 22, 2014

Problem 1 (20%) *Eigenvalues*

Consider

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad B = A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

a) (5%) Determine the eigenvalues, λ_1, λ_2 and eigenvectors of A.

b) (5%) Determine the eigenvalues and eigenvectors of B directly based on a).

c) (5%) Why is $\lambda_1^2 + \lambda_2^2 = 13$ in a)?

d) (5%) Show using A how an elementary matrix operation such as row exchange produces different eigenvalues.

Problem 2 (20%) *Eigenvalues of* AB

Consider

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

a) (10%) Find the eigenvalues and eigenvectors of A, B, AB, and BA.

b) (5%) Are the eigenvalues of AB equal to the eigenvalues of BA?

c) (5%) Are the eigenvalues of AB equal to the eigenvalues of A times the eigenvalues of B?

Problem 3 (20%) *Rank, Determinant, and Eigenvalues*

The 3×3 matrix B is known to have eigenvalues 0, 1, and 2. Is this information enough to determine the following (if yes, give the answer and explain; if no, explain why not):

a) (5%) the rank of B;

b) (5%) the determinant of $B^T B$;

c) (5%) the eigenvalues of $B^T B$;

d) (5%) the eigenvalues of $(B^2 + I)^{-1}$.

Please submit your homework either to ED 823 or to the TAs during the Thursday class time

Hint: B *is diagonalizable. Use the fact that*

$$\operatorname{rank}(\mathsf{AB}) \leq \min\{\operatorname{rank}(\mathsf{A}), \operatorname{rank}(\mathsf{B})\}. \qquad \text{(Tips in Slide 3-41)}$$

Problem 4 (30%) ***Finding Eigenvalues in Different Ways***

Consider the following six matrices:

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\mathsf{C} = \mathsf{A} - \mathsf{I} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \qquad \mathsf{D} = \mathsf{I} - \mathsf{A} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

$$\mathsf{E} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \qquad \mathsf{F} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Find the eigenvalues, rank and determinants of these six matrices.
Note: You do not have to stick to solving $\det(\mathsf{A} - \lambda\mathsf{I}) = 0$. Some quick tricks can be used.

Problem 5 (10%) ***Power of Matrix***

For $\mathsf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, prove that $\mathsf{A}^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$.