Linear Algebra

Homework 11 of May 29, 2014

Spring Semester 2014
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Problem 1 (60%) ***True or False***

Answer whether the following statements are true (with reason) or false (with counterexample). Notably, what is required in the answer is not a rigorous proof but simply a reason if the statement is true.

a) Suppose $\mathbf{x}$ is an eigenvector of A with eigenvalue $\lambda = 1$, i.e., $\mathsf{A}\mathbf{x} = \mathbf{x}$. Then $\lambda = -1$ is also an eigenvalue with eigenvector $-\mathbf{x}$.

b) It is possible to find a matrix A that has the diagonal elements being all zero, that is invertible, and that has only real eigenvalues.

c) If the columns of $\mathsf{S}_{n\times n}$ (whose columns are eigenvectors of A) are linearly independent, then

   i) A is invertible.
   ii) A is diagonalizable.
   iii) S is invertible.
   iv) S is diagonalizable.

d) The following statements are about matrix symmetry.

   i) If A has real eigenvalues and eigenvectors, then A is symmetric.
   ii) If $\mathsf{A}_{n\times n}$ has real eigenvalues and n orthogonal eigenvectors, then A is symmetric.
   iii) If an invertible A is symmetric, then A^{-1} is also symmetric.
   iv) If A is symmetric, then its eigen-matrix S is also symmetric.

e) The following statements are about positive-definiteness.

   i) If A is positive definite, then A^{-1} is also positive definite.
   ii) If A is positive definite, then A is invertible.
   iii) A projection matrix (i.e., $\mathsf{P}^2 = \mathsf{P}$) is positive definite.
   iv) A diagonal matrix with positive diagonal entries is positive definite.
   v) A symmetric matrix with a positive determinant is positive definite.

f) The following statements are about matrix similarity.

   i) A symmetric matrix cannot be similar to a non-symmetric matrix.
   ii) An invertible matrix cannot be similar to a singular matrix.
   iii) Matrix A cannot be similar to $-\mathsf{A}$ unless $\mathsf{A} = \mathsf{O}$, where O is the all-zero matrix.
   iv) Matrix A cannot be similar to $\mathsf{A} + \mathsf{I}$.
   v) If A is similar to B, then A^2 is similar to B^2.

Please submit your homework either to ED 823 or to the TAs during the Thursday class time

Problem 2 (10%) — *Eigenvectors of Symmetric Matrices are Perpendicular*

Here, we provide an alternative proof for the fact that eigenvectors are perpendicular to each other when A is symmetric.

a) (5%) Suppose that $\mathsf{A}\mathbf{x} = \lambda\mathbf{x}$ and $\mathsf{A}\mathbf{y} = 0\mathbf{y}$, where $\lambda \neq 0$. Argue that $\mathbf{x}$ is in the column space of A and that $\mathbf{y}$ is in the nullspace of A. Why are $\mathbf{x}$ and $\mathbf{y}$ perpendicular?

b) (5%) If $\mathsf{A}\mathbf{z} = \beta\mathbf{z}$ for some $\beta \neq \lambda$, apply the argument in a) to $(\mathsf{A} - \beta\mathsf{I})\mathbf{x} = (\lambda - \beta)\mathbf{x}$ and $(\mathsf{A} - \beta\mathsf{I})\mathbf{z} = 0\mathbf{z}$. Show that the eigenvectors with respect to distinct eigenvalues are perpendicular.

Problem 3 (20%) — *Test for Positive Definiteness*

a) (10%) Which of the following matrices only has positive eigenvalues? Do not compute the eigenvalues but use the test for positive definiteness. For those matrices that are not positive definite, find a vector $\mathbf{x}$ such that $\mathbf{x}^\mathsf{T}\mathsf{A}\mathbf{x} \leq 0$.

$$\mathsf{A}_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}, \qquad \mathsf{A}_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}, \qquad \mathsf{A}_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}, \qquad \mathsf{A}_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

b) (10%) For what numbers of b and c are the below two matrices positive definite?

$$\mathsf{A} = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}.$$

Factor each matrix into LDL^T.

Problem 4 (10%) — *Positive Definiteness*

Show that if the columns of $\mathsf{A}_{n\times n}$ are linearly independent and $\mathsf{C}_{n\times n}$ is positive definite, then $\mathsf{A}^\mathsf{T}\mathsf{C}\mathsf{A}$ is also positive definite.
Hint: $\mathbf{x}^\mathsf{T}\mathsf{A}^\mathsf{T}\mathsf{C}\mathsf{A}\mathbf{x}$.