

Chapter 5

Determinants

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5.1 The properties of determinants

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- Recall for Gauss-Jordan method and square A , $A\mathbf{x} = \mathbf{b}$ has a unique solution if **all pivots** are non-zero.

Note:

- A set of numbers are non-zeros if, and only if, their product is non-zero!
- $A\mathbf{x} = \mathbf{b}$ has a unique solution if, and only if, the **product of all pivots** is non-zero.

- Hence, the product of all pivots is an important **decisive factor** for a linear equation $A\mathbf{x} = \mathbf{b}$ (or for a matrix A).

Definition (Determinant): The determinant of a **square** matrix A , denoted by $\det(A)$, is equal to the product of all pivots, multiplying by $(-1)^m$, where m is the number of row exchanges performed during the derivation of these pivots.

Note: $\det(A)$ is sometimes denoted by $|A|$ (This notation is identical to “take the absolute value”); however, it needs to be pointed out that the latter notation may be a little misleading because it is possible that $\det(A) = |A| < 0$.

5.1 The properties of determinants

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- Alternative calculations of $\det(A)$:

– **Pivot formula:** $\det(A)$ = product of all pivots (if no row exchanges are performed in the process of producing pivots)

– **Leibniz formula:**

$$\det(A) = \sum_{\sigma \in \mathbf{P}_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$$

where \mathbf{P}_n is the set of all permutation of $(1, 2, \dots, n)$, and

$$\text{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ can be recovered to } (1, 2, \dots, n) \text{ by even number} \\ & \text{of pair-wise switching} \\ -1 & \text{otherwise} \end{cases}$$

Example.

$$\begin{aligned} \det(A_{3 \times 3}) &= (-1)^0 a_{1,1} a_{2,2} a_{3,3} + (-1)^1 a_{1,1} a_{2,3} a_{3,2} + (-1)^1 a_{1,2} a_{2,1} a_{3,3} \\ &+ (-1)^2 a_{1,2} a_{2,3} a_{3,1} + (-1)^2 a_{1,3} a_{2,1} a_{3,2} + (-1)^1 a_{1,3} a_{2,2} a_{3,1} \end{aligned}$$

5.1 The properties of determinants

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– **Leibniz formula (continued):**

- * Each product in the Leibniz formula takes exactly one entry from each row (analogously, one entry from each column).
- * So, there are $n!$ products in the Leibniz formula. (There are n selections from the 1st row, $(n - 1)$ selections from the 2nd row, $(n - 2)$ selections from the 3rd row, \dots , etc.)

– **Co-factor formula:**

$$\det(A) = \sum_{j=1}^n a_{\ell,j} C_{\ell,j}$$

where the co-factor $C_{\ell,j}$ is equal to $(-1)^{\ell+j}$ multiplying the determinant of the matrix that results from A by removing the ℓ -th row and the j -th column.

- The proofs of equivalences of these formulas will be given later.
- We will base on the Pivot formula (by its definition) to show ten fundamental properties of determinant.

5.1 Properties of determinant

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Ten properties listed in the textbook (first only proved for row operations; their validity for column operations will be given in Property 10)

1. Unit determinant for identity matrix. (Hint: By Pivot formula or by definition.)
2. Sign reversal by row/column switch. (Hint: By Pivot formula.)

By Pivot formula, a row switch matrix \tilde{A} of A simply results in different permutation matrix \tilde{P} to satisfy $PA = \tilde{P}\tilde{A} = LU$. The property follows since \tilde{P} gives one more row exchange during the LU decomposition.

5.1 Properties of determinant

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3. Linearity with respect to any row/column. (Hint: By Pivot formula.)

- Multiplication: $\det \begin{pmatrix} t \cdot a_{1,1} & t \cdot a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = t \cdot \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$

- Addition:

$$\det \begin{pmatrix} a_{1,1} + a'_{1,1} & a_{1,2} + a'_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} + \det \begin{pmatrix} a'_{1,1} & a'_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

Note: $\det(t \cdot A) = t^n \cdot \det(A) \neq t \cdot \det(A)$ for $t \neq 1$. As an additional note,

$$t \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} = \begin{bmatrix} t \cdot a_{1,1} & t \cdot a_{1,2} \\ t \cdot a_{2,1} & t \cdot a_{2,2} \end{bmatrix} \neq \begin{bmatrix} t \cdot a_{1,1} & t \cdot a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix}.$$

Recall that when the rows of A are edges of a box, the absolute value of $\det(A)$ gives the “volume” of the box! So, extending each edge by t -fold will increase the “volume” by t^n -fold.

5.1 Properties of determinant

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4. Two equivalent rows/columns make zero determinant.

Can be proved by Property 2 since $\det(A) = -\det(A)$ implies $\det(A) = 0$.

5. Subtracting one row (column) by a multiple of the other row (column) makes no change on determinant.

This property can be proved by Properties 3 and 4.

$$\det \left(\begin{bmatrix} a_{1,1} + t \cdot a_{2,1} & a_{1,2} + t \cdot a_{2,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \right) = \det \left(\begin{bmatrix} a_{1,1} & a_{2,2} \\ a_{1,2} & a_{2,2} \end{bmatrix} \right) + \underbrace{t \cdot \det \left(\begin{bmatrix} a_{2,1} & a_{2,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \right)}_{=0}$$

- So the Gauss-Jordan method (or more specifically, forward elimination) will not change the determinant.

6. Zero rows/columns give zero determinant.

This property can be proved by Property 3.

$$\det \left(\begin{bmatrix} 0 \cdot a_{1,1} & 0 \cdot a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \right) = 0 \cdot \det \left(\begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \right)$$

5.1 Properties of determinant

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7. Determinant of a triangular matrix is the product of diagonal terms.

This property holds by Pivot formula or by definition.

Also, from Property 5, we know that the forward elimination can be used to determine the determinant as

$$\det(A) = \det(U) = \prod_{i=1}^n u_{i,i}.$$

Note: If there are odd number of row exchanges performed during forward elimination, then

$$\det(A) = -\det(U).$$

5.1 Properties of determinant

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8. $\det(A) \neq 0$ if, and only if, A is invertible.

This property follows from $\det(A) = \det(U)$, i.e., from the discussion in Property 7.

Equivalently, we can state that $\det(A) = 0$ if, and only if, A is singular (or non-invertible).

5.1 Properties of determinant

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9. $\det(AB) = \det(A) \cdot \det(B)$

Proof:

- We first show that $\det(EB) = \det(E) \cdot \det(B)$ for any elementary matrix E , where an elementary matrix is one that performs
 - 1) row-pair combination (i.e., new row $j = \text{old row } j - c_{j,i} \times \text{row } i$),
 - 2) multiplying one row (i.e., new row $j = c_{j,j} \times \text{old row } j$), and
 - 3) row switching.
- As a result, the E that makes valid $\det(EB) = \det(E) \cdot \det(B)$ can be obtained from the identity matrix I by
 - 1) replacing (j, i) th element by $-c_{j,i}$; (Property 5)
 - 2) replacing (j, j) th element by $c_{j,j}$; (Property 3: Multiplication with $t = c_{j,j}$)
 - 3) switching the respective two rows. (Property 2)
- Now suppose one of $\det(A)$ and $\det(B)$ is zero. So, by Property 8, one of A and B is not invertible, so is AB ; thus, $\det(AB) = \det(A)\det(B) = 0$.

5.1 Properties of determinant

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Proof (cont):

- Now suppose both $\det(A)$ and $\det(B)$ are not zero. Then, by Gauss-Jordan method (or **rref**), we can respectively decompose A and B into

$$A = E_1 E_2 \cdots E_k \quad \text{and} \quad B = E'_1 E'_2 \cdots E'_{k'}.$$

(Exercise: What is **rref**(A) if $\det(A) \neq 0$?)

As a result,

$$\begin{aligned} \det(AB) &= \det(E_1 E_2 \cdots E_k E'_1 E'_2 \cdots E'_{k'}) \\ &= \det(E_1) \det(E_2) \cdots \det(E_k) \det(E'_1) \det(E'_2) \cdots \det(E'_{k'}) \\ &= \det(E_1 E_2 \cdots E_k) \det(E'_1 E'_2 \cdots E'_{k'}) \\ &= \det(A) \det(B). \end{aligned}$$

□

Note: The formal proof above is exactly what has stated in the textbook (that gives the idea behind the proof in an informal way).

5.1 Properties of determinant

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10. $\det(A) = \det(A^T)$

This property can be proved by $A = E_1 E_2 \cdots E_k$ and $\det(E_j) = \det(E_j^T)$ for every j .

5.1 A powerful tool: Elementary matrix

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The elementary matrix E can be obtained from the identity matrix I by

- 1) replacing (j, i) th element by $-c_{j,i}$;

In such case, $\det(E) = 1$. (Proved by Property 5)

- 2) replacing (j, j) th element by $c_{j,j}$;

In such case, $\det(E) = c_{j,j}$. (Proved by Property 3: Multiplication with $t = c_{j,j}$)

- 3) switching the respective two rows.

In such case, $\det(E) = -1$. (Proved by Property 2)

So,

$$\det(A) = \det(E_1)\det(E_2) \cdots \det(E_k) = (+1)^{\#1}(-1)^{\#3} \prod_{j=1}^n c_{j,j}$$

5.1 Eigenvalues

Problems 22 and 23:

(Problem 22, Section 5.1) From $ad - bc$, find the determinant of A and A^{-1} and $A - \lambda I$:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix} \quad \text{and} \quad A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{bmatrix}$$

Which two numbers λ lead to $\mathbf{det}(A - \lambda I) = 0$? Write down the matrix $A - \lambda I$ for each of those numbers λ — it should not be invertible.

(Problem 23, Section 5.1) From $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ find A^2 and A^{-1} and $A - \lambda I$ and their determinants. Which two numbers λ lead to $\mathbf{det}(A - \lambda I) = 0$?

- The **eigenvalues** of a matrix A is the solutions λ of $\mathbf{det}(A - \lambda I) = 0$.
- This is important because for these λ 's, there exists a non-zero vector \mathbf{v} such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Equivalently, there exists a non-zero vector satisfying $(A - \lambda I)\mathbf{v} = 0$.

Vector \mathbf{v} is usually referred to as the **eigenvector** associated with **eigenvalue** λ .

5.1 Co-factor

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Problem 30 (also, Problem 27 in Section 5.2):

(Problem 30, Section 5.1) (Calculus question) Show that the partial derivatives of $\ln(\det A)$ give A^{-1} !

$$f(a, b, c, d) = \ln(ad - bc) \quad \text{leads to} \quad \begin{bmatrix} \partial f / \partial a & \partial f / \partial c \\ \partial f / \partial b & \partial f / \partial d \end{bmatrix}.$$

(Problem 27, Section 5.2) (A calculus question) Show that the derivative of $\det A$ with respect to $a_{1,1}$ is the cofactor $C_{1,1}$. The other entries are fixed — we are only changing $a_{1,1}$.

- Let's look at the clue of the answers of these two problems first.
- We will come back to give a formal proof about “cofactor matrix can be used to provide the inverse of A ” later in Section 5.2.

5.1 Co-factor

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- Can the Co-factor matrix be used to provide the **inverse** of a matrix A ?

Direct answer: Yes.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln \det(A)}{\partial a_{1,1}} & \frac{\partial \ln \det(A)}{\partial a_{2,1}} & \cdots & \frac{\partial \ln \det(A)}{\partial a_{n,1}} \\ \frac{\partial \ln \det(A)}{\partial a_{1,2}} & \frac{\partial \ln \det(A)}{\partial a_{2,2}} & \cdots & \frac{\partial \ln \det(A)}{\partial a_{n,2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \ln \det(A)}{\partial a_{1,n}} & \frac{\partial \ln \det(A)}{\partial a_{2,n}} & \cdots & \frac{\partial \ln \det(A)}{\partial a_{n,n}} \end{bmatrix}$$

where the first equality (blue-colored part) is directly given without proof, and the second equality follows from the co-factor formula below.

– **Co-factor formula:**

$$\det(A) = \sum_{j=1}^n a_{i,j} C_{i,j}$$

where the co-factor $C_{i,j}$ is equal to $(-1)^{i+j}$ multiplying the determinant of the matrix that results from A by removing the i -th row and the j -th column.

5.1 Co-factor

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Problem 11 in Section 5.2:

(Problem 11, Section 5.2) Find all cofactors and put them into cofactor matrices C , D . Find AC AC^T and $\det B$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

Puzzle asked in the solution: $\det B = -21$ and $\det D = (-21)^2$. *Why?* See the next page.

– $C = \begin{bmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,n} \end{bmatrix}$ is called the **cofactor matrix** of A .

– C^T is called the **adjugate matrix** of A , denoted by $\text{adj}(A)$.

– So,

$$A^{-1} = \frac{1}{\det(A)} C^T.$$

– Hence in Problem 11, $AC^T = \det(A) \cdot I$ but $AC = \det(A) \cdot A(A^{-1})^T$.

– The solution given on page 535 of the textbook for C should be C^T .

5.2 Inverse and co-factors

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Lemma: Suppose A is invertible. Then, $AC^T = \det(A)I$.

Proof:

- By following the co-factor formula, the diagonal terms of AC^T are all $\det(A)$.
- When inner-producting the j th row $(a_{j,1}, a_{j,2}, \dots, a_{j,n})$ with the co-factors of i th row $(C_{i,1}, C_{i,2}, \dots, C_{i,n})$, where $i \neq j$, we can replace the i th row by the j -th row to form a new matrix A^* , and then, $\det(A^*) = 0$ since two rows are equal. Then the co-factors of i -th row of A^* are exactly the same as the co-factors of i -th row of A , and the i -th row of A^* is $(a_{j,1}, a_{j,2}, \dots, a_{j,n})$. Hence,

$$\begin{aligned}\det(A^*) &= a_{i,1}^* C_{i,1} + a_{i,2}^* C_{i,2} + \dots + a_{i,n}^* C_{i,n} \\ &= a_{j,1} C_{i,1} + a_{j,2} C_{i,2} + \dots + a_{j,n} C_{i,n} = 0.\end{aligned}$$

□

For example, when $n = 3$, the lemma says

$$\begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} C_{1,1} & C_{2,1} & C_{3,1} \\ C_{1,2} & C_{2,2} & C_{3,2} \\ C_{1,3} & C_{2,3} & C_{3,3} \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix}$$

This answers the puzzle in Problem 11, Section 5.2.

5.2 Pivots and submatrices

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- An interesting property based on the pivot formula:

– Define

$$A_1 = [a_{1,1}], \quad \dots \quad A_k = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,k} \end{bmatrix}, \quad \dots \quad A_n = \begin{bmatrix} a_{1,1} & \cdots & a_{1,k} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ a_{k,1} & \cdots & a_{k,k} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & \cdots & \cdots & a_{n,n} \end{bmatrix}$$

- Suppose the LU decomposition of A_n gives pivots from forward elimination without row exchanges:

$$A_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ l_{2,1} & 1 & 0 & \cdots & 0 \\ l_{3,1} & l_{3,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n,1} & l_{n,2} & l_{n,3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} d_1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & d_2 & u_{2,3} & \cdots & d_{2,n} \\ 0 & 0 & d_3 & \cdots & d_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

– Then,

$$\det(A_k) = d_1 d_2 \cdots d_k.$$

- Idea behind the proof: Pivot d_k only depends on those elements in A_k .

5.2 Leibniz formula revisited

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- In Section 5.1, we define and prove ten properties of determinants based on the **Pivot formula**. We now show that the **Pivot formula** is equivalent to the **Leibniz formula** by these properties.

$$\det(A) = \sum_{\sigma \in P_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}.$$

- By **Property 3**,

$$\begin{aligned} & \det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \\ &= \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} + \det \begin{pmatrix} 0 & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} + \det \begin{pmatrix} 0 & 0 & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \\ & \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ a_{2,1} & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} + \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} + \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \\ & \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & 0 & 0 \end{pmatrix} + \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & a_{3,2} & 0 \end{pmatrix} + \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & 0 & a_{3,3} \end{pmatrix} \end{aligned}$$

5.2 Leibniz formula revisited

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$$\begin{aligned}
 &= \dots \\
 &= \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix} + \det \begin{pmatrix} 0 & a_{1,2} & 0 \\ 0 & 0 & a_{2,3} \\ a_{3,1} & 0 & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & 0 & a_{1,3} \\ a_{2,1} & 0 & 0 \\ 0 & a_{3,2} & 0 \end{pmatrix} \\
 &+ \det \begin{pmatrix} a_{1,1} & 0 & 0 \\ 0 & 0 & a_{2,3} \\ 0 & a_{3,2} & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & a_{1,2} & 0 \\ a_{2,1} & 0 & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix} + \det \begin{pmatrix} 0 & 0 & a_{1,3} \\ 0 & a_{2,2} & 0 \\ a_{3,1} & 0 & 0 \end{pmatrix} \\
 &= a_{1,1}a_{2,2}a_{3,3} \cdot \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + a_{1,2}a_{2,3}a_{3,1} \cdot \det \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + a_{1,3}a_{2,1}a_{3,2} \cdot \det \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
 &+ a_{1,1}a_{2,3}a_{3,2} \cdot \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + a_{1,2}a_{2,1}a_{3,3} \cdot \det \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + a_{1,3}a_{2,2}a_{3,1} \cdot \det \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
 &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} - a_{1,3}a_{2,2}a_{3,1}
 \end{aligned}$$

5.2 Leibniz formula revisited

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The usefulness of the Leibniz formula can be seen in the below example. (So, it is called **the big formula** in the textbook.)

Example. Find the determinant of $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Solution.

- The Leibniz formula needs to choose exactly one element from each row.
- The elements chosen from the remaining rows cannot be located at the same columns as the ones that have been chosen (because the Leibniz formula also needs to choose exactly one element from each column).
- So, the only nonzero elements at the 1st and 4th rows need to be chosen (otherwise the product will be zero); so,

$$\det(A) = \det \left(\begin{bmatrix} 0 & \mathbf{1} & 0 & 0 \\ 1 & - & - & 0 \\ 0 & - & - & 1 \\ 0 & 0 & \mathbf{1} & 0 \end{bmatrix} \right) = \operatorname{sgn}(2, 1, 4, 3) \cdot a_{1,2}a_{2,1}a_{3,4}a_{4,3} = 1.$$

□

5.2 Co-factor formula revisited

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The **co-factor formula** can be proved by the **Leibniz formula**.

Co-factor formula:

$$\det(A) = \sum_{j=1}^n a_{\ell,j} C_{\ell,j}$$

where the co-factor $C_{\ell,j}$ is equal to $(-1)^{\ell+j}$ multiplying the determinant of the matrix that results from A by removing the ℓ -th row and the j -th column.

$$\begin{aligned} \det(A_{3 \times 3}) &= (-1)^0 a_{1,1} a_{2,2} a_{3,3} + (-1)^1 a_{1,1} a_{2,3} a_{3,2} + (-1)^1 a_{1,2} a_{2,1} a_{3,3} \\ &\quad + (-1)^2 a_{1,2} a_{2,3} a_{3,1} + (-1)^2 a_{1,3} a_{2,1} a_{3,2} + (-1)^1 a_{1,3} a_{2,2} a_{3,1} \\ &= \operatorname{sgn}(1, 2, 3) a_{1,1} a_{2,2} a_{3,3} + \operatorname{sgn}(1, 3, 2) a_{1,1} a_{2,3} a_{3,2} + \operatorname{sgn}(2, 1, 3) a_{1,2} a_{2,1} a_{3,3} \\ &\quad + \operatorname{sgn}(2, 3, 1) a_{1,2} a_{2,3} a_{3,1} + \operatorname{sgn}(3, 1, 2) a_{1,3} a_{2,1} a_{3,2} + \operatorname{sgn}(3, 2, 1) a_{1,3} a_{2,2} a_{3,1} \\ &= a_{2,1} (\operatorname{sgn}(2, 1, 3) a_{1,2} a_{3,3} + \operatorname{sgn}(3, 1, 2) a_{1,3} a_{3,2}) \\ &\quad + a_{2,2} (\operatorname{sgn}(1, 2, 3) a_{1,1} a_{3,3} + \operatorname{sgn}(3, 2, 1) a_{1,3} a_{3,1}) \\ &\quad + a_{2,3} (\operatorname{sgn}(1, 3, 2) a_{1,1} a_{3,2} + \operatorname{sgn}(2, 3, 1) a_{1,2} a_{3,1}) \\ &= a_{2,1} (-1)^{2+1} (\operatorname{sgn}(2, 3) a_{1,2} a_{3,3} + \operatorname{sgn}(3, 2) a_{1,3} a_{3,2}) \\ &\quad + a_{2,2} (-1)^{2+2} (\operatorname{sgn}(1, 3) a_{1,1} a_{3,3} + \operatorname{sgn}(3, 1) a_{1,3} a_{3,1}) \\ &\quad + a_{2,3} (-1)^{2+3} (\operatorname{sgn}(1, 2) a_{1,1} a_{3,2} + \operatorname{sgn}(2, 1) a_{1,2} a_{3,1}) \\ &= a_{2,1} C_{2,1} + a_{2,2} C_{2,2} + a_{2,3} C_{2,3} \end{aligned}$$

5.2 Co-factor formula revisited

5-23

$$\begin{aligned}
 \det(A) &= \sum_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma(1), \sigma(2), \dots, \sigma(n)) \prod_{i=1}^n a_{i, \sigma(i)} \\
 &= \sum_{j=1}^n a_{\ell, j} \sum_{\sigma \in \mathcal{P}_n: \sigma(\ell)=j} \operatorname{sgn}(\sigma(1), \dots, \underbrace{\sigma(\ell)}_{=j}, \dots, \sigma(n)) \prod_{i=1, i \neq \ell}^n a_{i, \sigma(i)} \\
 &= \sum_{j=1}^n a_{\ell, j} (-1)^{\ell+j} \underbrace{\sum_{\sigma \in \mathcal{P}_n: \sigma(\ell)=j} \operatorname{sgn}(\sigma(1), \dots, \sigma(\ell-1), \sigma(\ell+1), \dots, \sigma(n)) \prod_{i=1, i \neq \ell}^n a_{i, \sigma(i)}}_{C_{\ell, j}}
 \end{aligned}$$

Example. Find the determinant of $A = \begin{bmatrix} 0 & a_{1,2} & 0 & 0 \\ a_{2,1} & 0 & a_{2,3} & 0 \\ 0 & a_{3,2} & 0 & a_{3,4} \\ 0 & 0 & a_{4,3} & 0 \end{bmatrix}$.

5.2 Co-factor formula revisited

5-24

Solution.

$$\begin{aligned}\det(A) &= \det \left(\begin{bmatrix} 0 & a_{1,2} & 0 & 0 \\ a_{2,1} & 0 & a_{2,3} & 0 \\ 0 & a_{3,2} & 0 & a_{3,4} \\ 0 & 0 & a_{4,3} & 0 \end{bmatrix} \right) \\ &= a_{1,2}(-1)^{1+2} \cdot \det \left(\begin{bmatrix} a_{2,1} & a_{2,3} & 0 \\ 0 & 0 & a_{3,4} \\ 0 & a_{4,3} & 0 \end{bmatrix} \right) = -a_{1,2} \cdot \det \left(\begin{bmatrix} a_{2,1} & a_{2,3} & 0 \\ 0 & 0 & a_{3,4} \\ 0 & a_{4,3} & 0 \end{bmatrix} \right) \\ &= a_{1,2}a_{3,4} \cdot \det \left(\begin{bmatrix} a_{2,1} & a_{2,3} \\ 0 & a_{4,3} \end{bmatrix} \right) = a_{1,2}a_{3,4}a_{2,1}a_{4,3}\end{aligned}$$

□

Correction:

Problem 14: “The matrices in Problem 15” (should be Problem 13).

Problem 18: “Go back to B_n to Problem 14” (should be Problem 17).

5.2 Co-factor formula revisited

5-25

(Problem 13, Section 5.2) The n by n determinant C_n has 1's above and below the main diagonal:

$$C_1 = |0| \quad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \quad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

- (a) What are these determinants C_1, C_2, C_3, C_4 ? Hint: Reduce C_4 to C_2 .
- (b) By cofactors find the relation between C_n and C_{n-1} and C_{n-2} . Find C_{10} .

(Problem 14, Section 5.2) The matrices in Problem ~~13~~ 13 have 1's just above and below the main diagonal. Going down the matrix, which order of columns (if any) gives all 1's? Explain why that permutation is *even* for $n = 4, 8, 12, \dots$ and *odd* for $n = 2, 6, 10, \dots$. Then

$$C_n = 0 \text{ (odd } n) \quad C_n = 1 \text{ (} n = 4, 8, \dots) \quad C_n = -1 \text{ (} n = 2, 6, \dots).$$

Hint: Permute the red-color 1's above. They give the only non-zero term in the Leibniz formula.

5.2 Co-factor formula revisited

5-26

(Problem 17, Section 5.2) The matrix B_n is the $-1, 2, -1$ matrix A_n except that $b_{1,1} = 1$ instead of $a_{1,1} = 2$. Using cofactors of the *last* row of B_4 show that $|B_4| = 2|B_3| - |B_2| = 1$.

$$B_4 = \left[\begin{array}{ccc|c} \mathbf{1} & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ \hline & & -1 & 2 \end{array} \right] \quad B_3 = \begin{bmatrix} \mathbf{1} & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix} \quad B_2 = \begin{bmatrix} \mathbf{1} & -1 \\ -1 & 2 \end{bmatrix}$$

The recursion $|B_n| = 2|B_{n-1}| - |B_{n-2}|$ is satisfied when every $|B_n| = 1$. This recursion is the same as for the A 's in Example 6. The difference is in the starting values 1, 1, 1 for the determinants of sizes $n = 1, 2, 3$.

(Problem 18, Section 5.2) Go back to B_n in Problem 17. It is the same as A_n except for $b_{1,1} = 1$. So use linearity in the first row, where $[1 \ -1 \ 0]$ equals $[2 \ -1 \ 0]$ minus $[1 \ 0 \ 0]$:

$$|B_n| = \begin{vmatrix} 1 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} = \begin{vmatrix} 2 & -1 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix} - \begin{vmatrix} 1 & 0 & & 0 \\ -1 & & & \\ & & A_{n-1} & \\ 0 & & & \end{vmatrix}$$

Linearity gives $|B_n| = |A_n| - |A_{n-1}|$. **Hint: $|B_n| = 1$ and $|A_n| = n + 1$.**

5.3 Cramer's rules, inverses, and volumns

5-27

- Now we know how to compute A^{-1} by the **co-factors** when A is a square matrix and $\det(A) \neq 0$.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix}$$

- We can then go back to solve $A\mathbf{x} = \mathbf{b}$ by this inverse, i.e.,

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

- Cramer however provides a different solutions based on **determinants** under the condition that $\det(A) \neq 0$.

5.3 Key idea of Cramer's rule

5-28

- Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the solution of $A\mathbf{x} = \mathbf{b}$.

- Then we can easily prove that

$$\begin{aligned} & \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{bmatrix} \end{aligned}$$

5.3 Key idea of Cramer's rule

5-29

- Then by the product rule,

$$\det(A) \cdot \det \begin{pmatrix} x_1 & 0 & 0 \\ x_2 & 1 & 0 \\ x_3 & 0 & 1 \end{pmatrix} = \det(A) \cdot x_1 = \det \begin{pmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{pmatrix}$$

- We finally obtain

$$x_1 = \frac{\det \begin{pmatrix} b_1 & a_{1,2} & a_{1,3} \\ b_2 & a_{2,2} & a_{2,3} \\ b_3 & a_{3,2} & a_{3,3} \end{pmatrix}}{\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}}$$

- Similarly,

$$x_2 = \frac{\det \begin{pmatrix} a_{1,1} & b_1 & a_{1,3} \\ a_{2,1} & b_2 & a_{2,3} \\ a_{3,1} & b_3 & a_{3,3} \end{pmatrix}}{\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}} \quad \text{and} \quad x_3 = \frac{\det \begin{pmatrix} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ a_{3,1} & a_{3,2} & b_3 \end{pmatrix}}{\det \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}}$$

5.3 Areas, volumes and determinants

5-30

Recall that all properties of determinants are derived from **the first three properties.**

1. Unit determinant for identity matrix. (Hint: By Pivot formula or by definition.)
2. Sign reversal by row/column switch. (Hint: By Pivot formula.)
3. Linearity with respect to any row/column. (Hint: By Pivot formula.)

- Multiplication: $\det \begin{pmatrix} ta_{1,1} & ta_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = t \cdot \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$

- Addition:

$$\det \begin{pmatrix} a_{1,1} + a'_{1,1} & a_{1,2} + a'_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} = \det \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} + \det \begin{pmatrix} a'_{1,1} & a'_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

As it turns out, all functions that satisfy these three properties should have the same value.

5.3 Areas, volumes and determinants

5-31

Example. Area of a parallelogram.

Solution. It is a function of two vectors \mathbf{v} and \mathbf{w} .

1. Unit area for perpendicular unit vectors.

$$f(\mathbf{v}, \mathbf{w}) = 1 \quad \text{if } \|\mathbf{v}\| = \|\mathbf{w}\| = 1 \text{ and } \mathbf{v} \cdot \mathbf{w} = 0.$$

2. Area remained unchanged by vector switch.

$$f(\mathbf{v}, \mathbf{w}) = f(\mathbf{w}, \mathbf{v}).$$

3. Linearity with respect to any vector.

- Multiplication: $f(t\mathbf{v}, \mathbf{w}) = t \cdot f(\mathbf{v}, \mathbf{w})$.
- Addition: $f(\mathbf{v} + \mathbf{v}', \mathbf{w}) = f(\mathbf{v}, \mathbf{w}) + f(\mathbf{v}', \mathbf{w})$. (See the next page.)

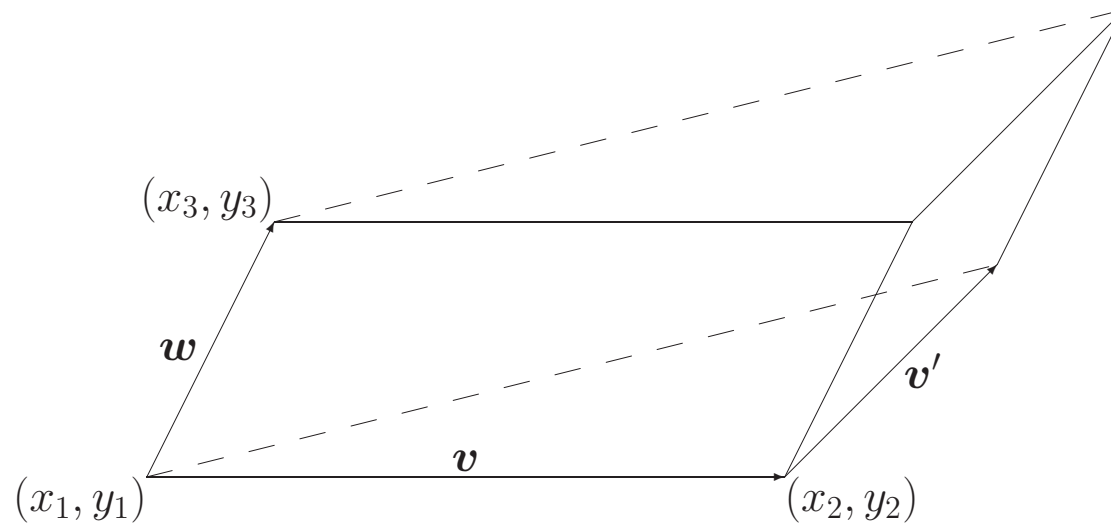
□

As it turns out, for two-dimensional vectors \mathbf{v} and \mathbf{w} ,

$$f(\mathbf{v}, \mathbf{w}) = \left| \det \left(\begin{bmatrix} \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix} \right) \right|.$$

5.3 Areas, volumes and determinants

5-32



- Three points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) also decide a parallelogram.
- Then the area of the parallelogram is

$$f(\mathbf{v}, \mathbf{w}) = \left| \det \begin{pmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{pmatrix} \right| = \left| \det \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \right|.$$

5.3 Areas, volumes and determinants

Similarly, for three-dimensional vectors \mathbf{u} , \mathbf{v} and \mathbf{w} ,

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \left| \det \begin{pmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{pmatrix} \right|$$

is the volume of the box spanned by the three vectors.

Again, the volume of the box decided by four vertices (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) and (x_4, y_4, z_4) can be written as

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \left| \det \begin{pmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \\ x_4 - x_1 & y_4 - x_1 & z_4 - z_1 \end{pmatrix} \right| = \left| \det \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \right|.$$

Note that the determinant formula does not in any case simplify the calculation of the area and volume. It is just interesting to have some alternative expression so that we can take advantage of it in some special case.

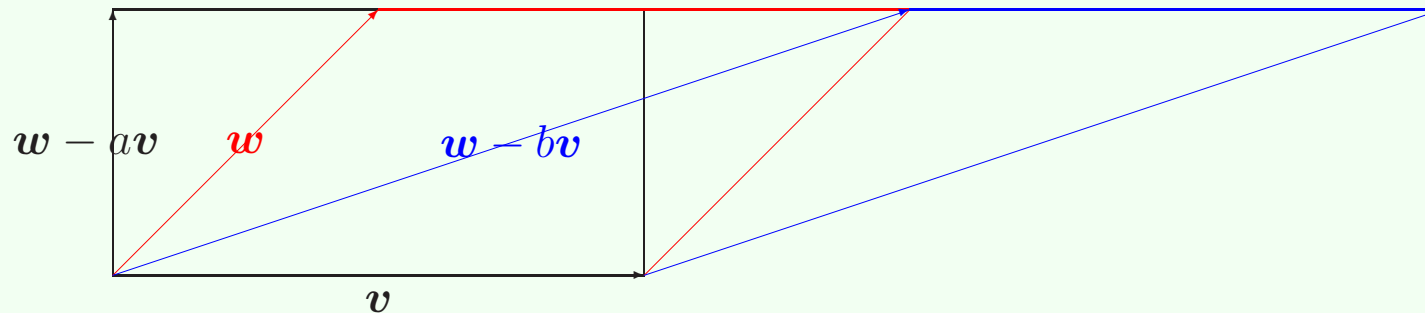
5.3 Areas, volumes and determinants

5-34

How to find the area of the parallelogram spanned by two (n -dimensional) vectors \mathbf{v} and \mathbf{w} , where $n > 2$? It might have no “determinant” formula. But we know we can apply, e.g., Property 5, to yield that

$$f(\mathbf{v}, \mathbf{w}) = f(\mathbf{v}, \mathbf{w} - a\mathbf{v})$$

such that $\mathbf{v} \cdot (\mathbf{w} - a\mathbf{v}) = 0$ (i.e., $a = \mathbf{v} \cdot \mathbf{w} / \|\mathbf{v}\|^2$).



(So all three parallelograms, i.e., the black one, the blue one and the red one, have the same area!)

Then,

$$f(\mathbf{v}, \mathbf{w}) = f(\mathbf{v}, \mathbf{w} - a\mathbf{v}) = \|\mathbf{v}\| \|\mathbf{w} - a\mathbf{v}\| = \|\mathbf{v}\| \left\| \mathbf{w} - \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|^2} \mathbf{v} \right\|.$$

5.3 Jacobian: An application of area computation

5-35

Example. Application of determinant computation.

- In calculus, we can transform **to** and **from** the 2-dimensional (x, y) -axis to the polar (r, θ) -axis with

$$\begin{cases} x(r, \theta) = r \cos(\theta) \\ y(r, \theta) = r \sin(\theta) \end{cases}$$

Linear approximation gives

$$\begin{cases} x(r + \Delta r, \theta + \Delta \theta) \approx x(r, \theta) + \frac{\partial x(r, \theta)}{\partial r} \Delta r + \frac{\partial x(r, \theta)}{\partial \theta} \Delta \theta \\ y(r + \Delta r, \theta + \Delta \theta) \approx y(r, \theta) + \frac{\partial y(r, \theta)}{\partial r} \Delta r + \frac{\partial y(r, \theta)}{\partial \theta} \Delta \theta \end{cases}$$

This approximation is accurate when Δr and $\Delta \theta$ get smaller.

So

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x(r + \Delta r, \theta + \Delta \theta) - x(r, \theta) \\ y(r + \Delta r, \theta + \Delta \theta) - y(r, \theta) \end{bmatrix} \approx \underbrace{\begin{bmatrix} \frac{\partial x(r, \theta)}{\partial r} & \frac{\partial x(r, \theta)}{\partial \theta} \\ \frac{\partial y(r, \theta)}{\partial r} & \frac{\partial y(r, \theta)}{\partial \theta} \end{bmatrix}}_{J(r, \theta)} \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix} = J(r, \theta) \begin{bmatrix} \Delta r \\ \Delta \theta \end{bmatrix}$$

5.3 Jacobian: An application of area computation

5-36

Now for the area of the (very small) parallelogram spanned by (x_1, y_1) , (x_2, y_2) and (x_3, y_3) , we have

$$\begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \approx J(r, \theta) \begin{bmatrix} r_2 - r_1 & r_3 - r_1 \\ \theta_2 - \theta_1 & \theta_3 - \theta_1 \end{bmatrix}$$

This gives the important result that for infinitesimal Δx and Δy (equivalently, Δr and $\Delta \theta$),

$$\begin{aligned} dA_{x,y} &= \left| \det \left(\begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \right) \right| \\ &= |\det(J(r, \theta))| \times \left| \det \left(\begin{bmatrix} r_2 - r_1 & r_3 - r_1 \\ \theta_2 - \theta_1 & \theta_3 - \theta_1 \end{bmatrix} \right) \right| \\ &= |\det(J(r, \theta))| dA_{r,\theta}. \end{aligned}$$

□

5.3 Cross product

5-37

- For two vectors \mathbf{v} and \mathbf{w} , we have defined the **inner product** and the **outer product** respectively as

$$\text{inner product: } \mathbf{v} \cdot \mathbf{w} = \mathbf{w}^T \mathbf{v} \quad \text{and} \quad \text{outer product: } \mathbf{v} \otimes \mathbf{w} = \mathbf{v} \mathbf{w}^T$$

for which the outputs are respectively a **scaler** and an $n \times n$ **matrix**.

- Now we introduce a third product: **cross product** for which the output is a **vector**.

$$\begin{aligned} \mathbf{v} \times \mathbf{w} &\triangleq \det \left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right) \\ &\triangleq \det \left(\begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} \right) \mathbf{i} - \det \left(\begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} \right) \mathbf{j} + \det \left(\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right) \mathbf{k}, \end{aligned}$$

where

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

5.3 Cross product

5-38

- You have noticed that the **cross product** for **3-dimensional vectors** is defined based on **co-factors**.
- This is no coincidence as the **idea** of the **cross product** is to output a **vector** that is orthogonal to both \mathbf{v} and \mathbf{w} .

$$\begin{aligned} & \mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) \\ &= \det \left(\begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} \right) \mathbf{v} \cdot \mathbf{i} - \det \left(\begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} \right) \mathbf{v} \cdot \mathbf{j} + \det \left(\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right) \mathbf{v} \cdot \mathbf{k} \\ &= \det \left(\begin{bmatrix} v_2 & v_3 \\ w_2 & w_3 \end{bmatrix} \right) v_1 - \det \left(\begin{bmatrix} v_1 & v_3 \\ w_1 & w_3 \end{bmatrix} \right) v_2 + \det \left(\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right) v_3 \\ &= \det \left(\begin{bmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} \right) = 0 \end{aligned}$$

- An extension definition for the **cross product** of higher dimensional vectors is based on similar idea (to output a proper vector that is orthogonal to the original two), but will be more complicated (in choosing what we mean by “proper.”)

5.3 Cross product and area

5-39

“proper” = the length of the vector should be the area spanned by the original two.

Example. Area of a parallelogram spanned by two 3-dimensional vectors \mathbf{v} and \mathbf{w} .

1. Unit area for perpendicular unit vectors.

$$\|\mathbf{v} \times \mathbf{w}\| = 1 \quad \text{if } \|\mathbf{v}\| = \|\mathbf{w}\| = 1 \text{ and } \mathbf{v} \cdot \mathbf{w} = 0.$$

Proof:

$$\begin{aligned} \|\mathbf{v} \times \mathbf{w}\|^2 &= (v_2w_3 + v_3w_2)^2 + (v_1w_3 + v_3w_1)^2 + (v_1w_2 + v_2w_1)^2 \\ &= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2) - (v_1w_1 + v_2w_2 + v_3w_3)^2 \\ &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 = 1. \end{aligned}$$

2. Area remained unchanged by vector switch.

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}.$$

3. Linearity with respect to any vector.

- Multiplication: $(t\mathbf{v}) \times \mathbf{w} = t(\mathbf{v} \times \mathbf{w})$.
- Addition: $(\mathbf{v} + \mathbf{v}') \times \mathbf{w} = \mathbf{v} \times \mathbf{w} + \mathbf{v}' \times \mathbf{w}$.

5.3 Cross product and area

5-40

□

As it turns out, for three-dimensional vectors \mathbf{v} and \mathbf{w} ,

$$\text{area}(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} \times \mathbf{w}\|.$$

Hence (as also can be known from

$$\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - (\mathbf{v} \cdot \mathbf{w})^2 = \|\mathbf{v}\|^2\|\mathbf{w}\|^2 - \|\mathbf{v}\|^2\|\mathbf{w}\|^2 \cos^2(\theta))$$

we have

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| |\sin(\theta)|.$$

5.3 Cross product and right-hand rule

- **Right-hand rule for cross product:** $\mathbf{v} \times \mathbf{w}$ points to the “thumb” when right hand fingers curl from \mathbf{v} to \mathbf{w} .

5.3 Triple product

5-41

Now we know **inner product**, **outer product**, **cross product**; let's introduce another **product of vectors** in the literature.

Definition (Triple product): The triple product of three vectors is defined as

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$$

- For 3-dimensional vectors,

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

gives the volume of the box spanned by \mathbf{u} , \mathbf{v} and \mathbf{w} .

5.3 A “potential” way to derive the nullspace

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Lemma: Suppose A is not invertible. Then,

$$AC^T = \text{all-zero matrix}$$

where C is the cofactor matrix.

Proof:

- The off-diagonal terms are zero by the same proof to show

$$AC^T = \det(A)I$$

for invertible A .

- The proof is completed by noting that the diagonal terms are equal to

$$\det(A) = 0.$$

□

5.3 A “potential” way to derive the nullspace

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- This seems to give an alternative way to derive the null space of A , i.e, all solutions to satisfy $A\mathbf{x} = \mathbf{0}$.

- However, when $A_{n \times n}$ has rank no greater than $(n - 2)$, we have

$$C_{i,j} = 0$$

for every i and j . So, this “alternative” method fails.

- This method works only when $A_{n \times n}$ has rank $(n - 1)$. Then, it is guaranteed that at least one column in C^T is non-zero.

5.3 Remarks

5-44

Problem 40. Jacobi's formula.

$$\frac{d}{dx} \det(A) = \text{trace} \left(\text{adj}(A) \frac{dA}{dx} \right)$$

(Problem 40, Section 5.3) Suppose A is a 5 by 5 matrix. Its entries in row 1 multiply determinants (cofactors) in row 2–5 to give the determinant. Can you guess “Jacobi formula” for $\det A$ using 2 by 2 determinants from rows 1-2 *times* 3 by 3 determinants from rows 3–5?

Test your formula on the $-1, 2, -1$ tridiagonal matrix that has determinant = 6.

Example of Jacobi's formula. $A = \begin{bmatrix} 2x^2 & x \\ x & 1 \end{bmatrix} \implies \det(A) = x^2.$

$$\text{adj}(A) = \text{transpose of cofactor matrix} = \begin{bmatrix} 1 & -x \\ -x & 2x^2 \end{bmatrix}$$

$$\frac{d}{dx} A = \begin{bmatrix} 4x & 1 \\ 1 & 0 \end{bmatrix}$$

$$\text{adj}(A) \frac{dA}{dx} = \begin{bmatrix} 1 & -x \\ -x & 2x^2 \end{bmatrix} \begin{bmatrix} 4x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3x & 1 \\ -2x^2 & -x \end{bmatrix}$$

$$\implies \frac{d}{dx} \det(A) = 2x = 3x + (-x) = \text{trace} \left(\text{adj}(A) \frac{dA}{dx} \right)$$

5.3 Remarks

5-45

The Jacobi's formula can be used to confirm the **co-factor formula** of the determinant.

$$\begin{aligned}
 \frac{d}{d a_{i,j}} \det(A) &= \text{trace} \left(\text{adj}(A) \frac{dA}{d a_{i,j}} \right) \\
 &= \text{trace} \left(\begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix} \underbrace{\begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}}_{1 \text{ in } (i,j) \text{th entry}} \right) \\
 &= \text{trace} \left(\underbrace{\begin{bmatrix} 0 & \cdots & C_{i,1} & \cdots & 0 \\ 0 & \cdots & \vdots & \cdots & 0 \\ 0 & \cdots & C_{i,j} & \cdots & 0 \\ 0 & \cdots & \vdots & \cdots & 0 \\ 0 & \cdots & C_{i,n} & \cdots & 0 \end{bmatrix}}_{j \text{th column}} \right) = C_{i,j}
 \end{aligned}$$

5.3 Remarks

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We can again apply the Jacobi's formula on $C_{i,j}$. With $k \neq i$ and $\ell \neq j$,

$$\frac{d}{d a_{k,\ell}} C_{i,j} = C_{(i,k),(j,\ell)}$$

where $C_{(i,k),(j,\ell)}$ is $(-1)^{k'+\ell'}$ multiplying the determinant $M_{(i,k),(j,\ell)}$ of $(n-2) \times (n-2)$ submatrix of A with i, k rows and k, ℓ columns removed, where

$$k' \triangleq \begin{cases} k & k < i \\ k - 1 & k > i \end{cases} \quad \text{and} \quad \ell' \triangleq \begin{cases} j & \ell < j \\ j - 1 & \ell > j \end{cases}.$$

So for $k > i$ and $\ell > j$

$$\begin{aligned} \det(A) &= \det \left(\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & a_{i,j} & \cdots & a_{i,\ell} & \cdots \\ \cdots & a_{k,j} & \cdots & a_{k,\ell} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \right) \\ &= \cdots + a_{i,j} a_{k,\ell} (-1)^{i+j} (-1)^{(k-1)+(\ell-1)} M_{(i,k),(j,\ell)} + a_{i,\ell} a_{k,j} (-1)^{i+\ell} (-1)^{k+(j-1)} M_{(i,k),(j,\ell)} + \cdots \\ &= \cdots + \det \left(\begin{bmatrix} a_{i,j} & a_{i,\ell} \\ a_{k,j} & a_{k,\ell} \end{bmatrix} \right) (-1)^{i+j+k+\ell} M_{(i,k),(j,\ell)} + \cdots \\ &= \sum_{j=1}^n \sum_{\ell=j+1}^n \det \left(\begin{bmatrix} a_{i,j} & a_{i,\ell} \\ a_{k,j} & a_{k,\ell} \end{bmatrix} \right) (-1)^{i+j+k+\ell} M_{(i,k),(j,\ell)}, \text{ an extension of the cofactor formula.} \end{aligned}$$

5.3 Remarks

5-47

Problem 41. Cauchy-Binet formula.

$$\det(A_{m \times n} B_{n \times m}) = \sum_{[m]} \det(A_{m \times [m]}) \cdot \det(B_{[m] \times m})$$

where $A_{m \times [m]}$ is the submatrix of A with only m columns selected from A , and $B_{[m] \times m}$ is the submatrix of B with only m rows selected from B .

The summation is carried out over all possible selections. There are $\binom{n}{m}$ of them.

(Problem 41, Section 5.3) The 2 by 2 matrix $AB = (2 \text{ by } 3)(3 \text{ by } 2)$ has a “Cauchy-Binet formula” for $\det AB$:

$$\det AB = \text{sum of (2 by 2 determinants in } A)(2 \text{ by 2 determinants in } B)$$

(a) Guess which 2 by 2 determinants to use from A and B .

(b) Test your formula when the rows of A are 1, 2, 3 and 1, 4, 7 with $B = A^T$.

5.3 Remarks

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Example. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}$ and $B = A^T$.

Solution.

$$\det(AB) = \det \left(\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 7 \end{bmatrix} \right) = \det \left(\begin{bmatrix} 14 & 30 \\ 30 & 66 \end{bmatrix} \right) = 24$$

$$\begin{aligned} \sum_{[2]} \det(A_{2 \times [2]}) \cdot \det(B_{[2] \times 2}) &= \det \left(\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \right) \det \left(\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \right) \\ &\quad + \det \left(\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix} \right) \det \left(\begin{bmatrix} 1 & 1 \\ 3 & 7 \end{bmatrix} \right) + \det \left(\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \right) \det \left(\begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix} \right) \\ &= 2 \cdot 2 + 4 \cdot 4 + 2 \cdot 2 = 24 \end{aligned}$$

□