Spring Semester 2014
Prof. Dr. Po-Ning Chen
http://shannon.cm.nctu.edu.tw/la.htm

Linear Algebra

Midterm Exam 1 of 27 March, 2014

Problem 1 (15%) ***Cauchy–Schwarz Inequality***

For positive real numbers a, b, c with $a + b + c = 1$, use Cauchy-Schwarz Inequality to show that

$$\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2 \geq \frac{100}{3}.$$

Hint: Use Cauchy-Schwarz Inequality

$$\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \geq \left|\mathbf{x}\cdot\mathbf{y}\right|^2$$

with $\mathbf{x} = \begin{bmatrix} a + \frac{1}{a} \\ b + \frac{1}{b} \\ c + \frac{1}{c} \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then use Cauchy-Schwarz inequality again to find a lower bound to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Solution . . . .

Denote for convenience

$$S = \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2.$$

Then Cauchy-Schwarz Inequality gives:

$$\begin{aligned} \|\mathbf{x}\|^2\|\mathbf{y}\|^2 &= \left(\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 + \left(c + \frac{1}{c}\right)^2\right)\left(1^2 + 1^2 + 1^2\right) = 3S \\ &\geq \left|\mathbf{x}\cdot\mathbf{y}\right|^2 = \left(a + \frac{1}{a} + b + \frac{1}{b} + c + \frac{1}{c}\right)^2, \end{aligned}$$

which implies

$$S \geq \frac{1}{3}\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2. \tag{1}$$

Thus, we need a lower bound to $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Set $\mathbf{v} = \begin{bmatrix} \sqrt{a} \\ \sqrt{b} \\ \sqrt{c} \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} \frac{1}{\sqrt{a}} \\ \frac{1}{\sqrt{b}} \\ \frac{1}{\sqrt{c}} \end{bmatrix}$. Then Cauchy-Schwarz Inequality $\|\mathbf{v}\|^2\|\mathbf{w}\|^2 \geq \left|\mathbf{v}\cdot\mathbf{w}\right|^2$ gives that

$$\left(\left(\sqrt{a}\right)^2 + \left(\sqrt{b}\right)^2 + \left(\sqrt{c}\right)^2\right)\left(\left(\frac{1}{\sqrt{a}}\right)^2 + \left(\frac{1}{\sqrt{b}}\right)^2 + \left(\frac{1}{\sqrt{c}}\right)^2\right) \geq (1 + 1 + 1)^2 = 3^2;$$

hence, we have

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9. \tag{2}$$

(??) and (??) together imply the desired result.

Problem 2 (15%) ***Right Inverse of a Non-Square Matrix***

Find the right inverse

$$\mathsf{R}_{4\times 3} = \begin{bmatrix} & \mathbf{r}_1' & \\ & \mathbf{r}_2' & \\ & \mathbf{r}_3' & \\ 1 & 1 & 1 \end{bmatrix}$$

of a non-square matrix

$$\mathsf{A}_{3\times 4} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix}$$

such that AR is equal to the 3 by 3 identity matrix $\mathsf{I}_{3\times 3}$.
Hint: Find rref(A).

Solution ....

Use forward eliminations to obtain rref(A): i.e., find E such that $\mathsf{EA} = \begin{bmatrix} \mathsf{I}_{3\times 3} & \mathsf{Ea}_4 \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathsf{Ea}_4 \end{bmatrix}$ with the following steps.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \mathsf{I}_{3\times 3} & \mathsf{Ea}_4 \end{bmatrix}.$$

We then compute the elimination matrix

$$\mathsf{E} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

In the end, refer to Problem 1 of Homework 3 to derive:

$$\begin{aligned} \mathsf{EAR} = \begin{bmatrix} \mathsf{I} & \mathsf{Ea}_4 \end{bmatrix} \begin{bmatrix} & \mathbf{r}_1' & \\ & \mathbf{r}_2' & \\ & \mathbf{r}_3' & \\ 1 & 1 & 1 \end{bmatrix} &= \mathbf{e}_1\mathbf{r}_1' + \mathbf{e}_2\mathbf{r}_2' + \mathbf{e}_3\mathbf{r}_3' + \mathsf{Ea}_4 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} + \mathsf{Ea}_4 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \\ &= \mathsf{E} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}. \end{aligned}$$

This leads to the solution that

$$\begin{bmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} - \mathsf{Ea}_4 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

and hence

$$\mathsf{R} = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 7 & 4 \\ -2 & -3 & -1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Problem 3 (12%) $\mathsf{PA} = \mathsf{LU}$ ***Factorization***

Find the $\mathsf{PA} = \mathsf{LU}$ factorization for

$$\mathsf{A} = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & 9 \\ 1 & 1 & 3 \end{bmatrix}.$$

Solution . . . .

We firstly exchange the first and the third rows; then do the forward eliminations for the resulting matrix:

$$\begin{aligned} \mathsf{P}_{1,3}\mathsf{A} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 \\ 0 & 2 & 9 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 9 \\ 0 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 6 \\ 0 & 1 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix} = \mathsf{LU}. \end{aligned}$$

Problem 4 (16%) ***Properties of Inverses***

True or False (with a counterexample if false and a proof if true):

a) (4%) If A is an invertible tridiagonal matrix, then A^{-1} is a tridiagonal matrix.

b) (4%) If A, B and C are all n by n matrices and $\mathsf{AB} + \mathsf{AC} = \mathsf{I}$, then A is invertible.

c) (4%) If both A and B are $n \times n$ and invertible, then $\mathsf{A} + \mathsf{B}$ is invertible.

d) (4%) If A is an invertible matrix and c is a non-zero real number, then $(c\mathsf{A})^{-1} = \frac{1}{c}\mathsf{A}^{-1}$.

Solution . . . .

a) False.

A counterexample is

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

of which the inverse is

$$\mathsf{A}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Hence, A^{-1} is not a tridiagonal matrix.

b) True.

Since $\mathsf{A}(\mathsf{B}+\mathsf{C}) = \mathsf{I}$, $\mathsf{A}^{-1} = \mathsf{B}+\mathsf{C}$.

c) False.

A counterexample is

$$\mathsf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \text{and} \qquad \mathsf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$\mathsf{A}+\mathsf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Since $\mathsf{A}+\mathsf{B}$ is not an invertible matrix, the statement is false.

d) True.

Multiplying $\frac{1}{c}$ on both sides of

$$(c\mathsf{A})(c\mathsf{A})^{-1} = \mathsf{I}$$

yields

$$(\frac{1}{c})(c\mathsf{A})(c\mathsf{A})^{-1} = \mathsf{A}(c\mathsf{A})^{-1} = \frac{1}{c}\mathsf{I}.$$

Then left-multiplying A^{-1} on both sides, we obtain

$$\mathsf{A}^{-1}\mathsf{A}(c\mathsf{A})^{-1} = \mathsf{A}^{-1}\frac{1}{c}\mathsf{I}$$
$$(c\mathsf{A})^{-1} = \frac{1}{c}\mathsf{A}^{-1}.$$

Problem 5 (18%) ***Inverse of Matrix***

a) (12%) Find the inverse of the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$

b) (6%) Solve the linear equations

$$x + 2y + 3z = 0$$
$$3x + 2y + 2z = 2$$
$$2x + y + z = -1.$$

Solution . . . .

a) The inverse of A is

$$A^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ -1 & 5 & -7 \\ 1 & -3 & 4 \end{bmatrix}.$$

b) Since

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 17 \\ -10 \end{bmatrix}.$$

Problem 6 (24%) UL ***Factorization***

Give a matrix A as

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

a) (12%) Solve $A = UL$ factorization, where U is an upper triangular matrix and L is an lower triangular matrix.

Hint: Do "backward" elimination first instead of "forward" elimination.

b) (6%) Let

$$\boldsymbol{b} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}.$$

Solve $U\boldsymbol{c} = \boldsymbol{b}$.

c) (6%) Solve $A\boldsymbol{x} = \boldsymbol{b}$.

Solution . . . .

a) Since

$$A = \begin{bmatrix} 1 & 4 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \xrightarrow{\text{backward elimination}} \begin{bmatrix} 4 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = UL.$$

b) Solving

$$\mathsf{U}\boldsymbol{c} = \boldsymbol{b} \Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{c} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix},$$

we get

$$\boldsymbol{c} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}.$$

c)

$$\begin{aligned} &\mathsf{A}\boldsymbol{x} = \boldsymbol{b} \\ \Rightarrow\ &\mathsf{UL}\boldsymbol{x} = \boldsymbol{b} \\ \Rightarrow\ &\mathsf{U}\boldsymbol{c} = \boldsymbol{b} \quad \text{if } \mathsf{L}\boldsymbol{x} = \boldsymbol{c}. \end{aligned}$$

Then solve $\mathsf{L}\boldsymbol{x} = \boldsymbol{c}$ and get

$$\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$