LINEAR ALGEBRA

Spring Semester 2014
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http://shannon.cm.nctu.edu.tw/la.htm

Midterm Exam 2 of 17 April, 2014

Problem 1 ***Finding a Basis***

Find a basis for each of the four subspaces of A below:

$$\mathsf{A} = \tilde{\mathsf{E}}\mathsf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & 2 & 1 & 0 \\ 7 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) (6%) The column space of A.

b) (6%) The row space of A.

c) (6%) The nullspace of A.

d) (6%) The left nullspace of A.

Problem 2 (8%) ***Extension to Matrix Space***

Determine all the 2×3 matrices whose nullspace is spanned by $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Hint: Treating $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathsf{F}_{2\times 1} \\ \mathsf{I}_{1\times 1} \end{bmatrix}$, you should be able to find a basis of the common row space of these matrices.

Problem 3 ***Subspaces***

Which of the following are subspaces? (With a counterexample if it is not and a reason if it is)

a) (6%) All the vectors $\boldsymbol{x}$ in $\mathbb{R}^3$ such that $\boldsymbol{x}^{\mathrm{T}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 0$.

b) (6%) All the vectors (x, y) in $\mathbb{R}^2$ such that $x^2 - y^2 = 0$.

c) (6%) All the vectors (x, y) in $\mathbb{R}^2$ such that $x + y = 1$.

d) (6%) All the vectors $\boldsymbol{x}$ in $\mathbb{R}^3$, which are in the column space OR in the nullspace (or in both) of matrix $\begin{bmatrix} 1 & 2 & -3 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{bmatrix}$.

Problem 4 ***Complete Solution***

Consider the matrix

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ -1 & -2 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

a) (8%) Find the complete solution of $\mathsf{A}\boldsymbol{x} = \mathbf{0}$.

b) (8%) Find the complete solution of $\mathsf{A}\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Problem 5 ***Rank of Matrix***

Let matrix A be m by n and matrix B be n by m. Suppose that $\mathsf{AB} = \mathsf{I}_{m\times m}$.

a) (6%) Let r denote the rank of the matrix A. What is the general relation between r and m (i.e., $r \geq m$, or $r \leq m$, or $r = m$)? You should justify your answer.

b) (6%) Which one could we conclude, $m \leq n$ or $m \geq n$? Why?

Problem 6 ***Rank of Matrix***

Let the vector space $\mathbb{V}$ consist of the following eight vectors:

$$\mathbb{V} \triangleq \left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix}, \begin{bmatrix} b \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \\ b \end{bmatrix} \right\},$$

where a, b are two symbols. Define the vector-to-vector addition as component-wise addition through

$$a + a = a, \quad a + b = b, \quad b + a = b, \quad b + b = a.$$

For example,

$$\begin{bmatrix} b \\ a \\ a \end{bmatrix} + \begin{bmatrix} b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \\ b \end{bmatrix}$$

Define the scaler-to-vector multiplication as:

$$a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \quad b \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

for any $x_1, x_2, x_3 \in \{a, b\}$. Let $\mathsf{A} = \begin{bmatrix} b & a \\ b & a \\ a & b \end{bmatrix}$. Answer the following questions.

a) (8%) List all the vectors in the column space $\boldsymbol{C}(\mathsf{A})$ of A.

Hint: Linear combinations of the two columns in A.

b) (8%) Determine the null space $\boldsymbol{N}(\mathsf{A})$ of A.

Hint: The nullspace of A consists of all the solutions $\mathsf{A}\boldsymbol{x} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$.