Linear Algebra

Midterm Exam 2 of 17 April, 2014

Spring Semester 2014
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Problem 1 ***Finding a Basis***

Find a basis for each of the four subspaces of A below:

$$\mathsf{A} = \tilde{\mathsf{E}}\mathsf{U} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & 2 & 1 & 0 \\ 7 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) (6%) The column space of A.

b) (6%) The row space of A.

c) (6%) The nullspace of A.

d) (6%) The left nullspace of A.

Solution ....

a) The column space basis of A can be chosen as the three pivot columns of A. By denoting

$$\mathsf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & 2 & 1 & 0 \\ 7 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \triangleq [\tilde{\mathbf{e}}_1 \quad \tilde{\mathbf{e}}_2 \quad \tilde{\mathbf{e}}_3 \quad \tilde{\mathbf{e}}_4] \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the three pivot columns of A are equal to

$$\tilde{\mathbf{e}}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}, \; 2\tilde{\mathbf{e}}_1 + 1\tilde{\mathbf{e}}_2 = \begin{bmatrix} 2 \\ 7 \\ 12 \\ 18 \end{bmatrix}, \; 3\tilde{\mathbf{e}}_1 + 2\tilde{\mathbf{e}}_2 + 1\tilde{\mathbf{e}}_3 = \begin{bmatrix} 3 \\ 11 \\ 20 \\ 32 \end{bmatrix}.$$

b) The row space basis can be chosen as the three pivot rows of U. So a quick answer is $\begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

c) Since the nullspace of A is identical to the nullspace of U, we compute $\mathsf{R} = \text{rref}(\mathsf{U})$ as

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{R}.$$

The basis can be obtained from the two columns of

$$\mathsf{N} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & -2 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}.$$

d) Since $\mathsf{A} = \tilde{\mathsf{E}}\mathsf{U}$, we have

$$\tilde{\mathsf{E}}^{-1}\mathsf{A} \triangleq \mathsf{E}\mathsf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 2 & 2 & -3 & 1 \end{bmatrix} \mathsf{A} = \begin{bmatrix} \mathsf{E}_{3\times 4} \\ \mathbf{e}_4^{\mathsf{T}} \end{bmatrix} \mathsf{A}_{4\times 5} = \begin{bmatrix} \mathsf{E}_{3\times 4}\mathsf{A}_{4\times 5} \\ \mathbf{e}_4^{\mathsf{T}}\mathsf{A}_{4\times 5} \end{bmatrix} = \mathsf{U} = \begin{bmatrix} 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Longrightarrow \mathbf{e}_4^{\mathsf{T}}\mathsf{A} = \mathbf{0} \Longrightarrow \mathsf{A}^{\mathsf{T}}\mathbf{e}_4 = \mathbf{0}.$$

Therefore, $\mathbf{e}_4 = \begin{bmatrix} 2 \\ 2 \\ -3 \\ 1 \end{bmatrix}$, i.e., the last row of E, can be a basis of the left nullspace of A.

Problem 2 (8%) — *Extension to Matrix Space*

Determine all the 2×3 matrices whose nullspace is spanned by $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Hint: Treating $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathsf{F}_{2\times 1} \\ \mathsf{I}_{1\times 1} \end{bmatrix}$, you should be able to find a basis of the common row space of these matrices.

Solution ....

The problem is the same as finding all the matrices whose row space is spanned by the two (linearly independent) vectors perpendicular to $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathsf{F}_{2\times 1} \\ \mathsf{I}_{1\times 1} \end{bmatrix}$. So one choice of the bases of the row space is given by

$$\begin{bmatrix} \mathsf{I}_{2\times 2} & \mathsf{F}_{2\times 1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}.$$

Therefore, all 2×3 matrices whose rows are linear combinations of the above two row vectors give the solution, *i.e.*,

$$\mathsf{A}_{2\times 3} = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}_{2\times 3},$$

where the two rows of A are not multiples of each other.

Problem 3 — *Subspaces*

Which of the following are subspaces? (With a counterexample if it is not and a reason if it is)

a) (6%) All the vectors $\boldsymbol{x}$ in $\mathbb{R}^3$ such that $\boldsymbol{x}^{\mathrm{T}}\begin{bmatrix}1\\2\\3\end{bmatrix} = 0$.

b) (6%) All the vectors (x, y) in $\mathbb{R}^2$ such that $x^2 - y^2 = 0$.

c) (6%) All the vectors (x, y) in $\mathbb{R}^2$ such that $x + y = 1$.

d) (6%) All the vectors $\boldsymbol{x}$ in $\mathbb{R}^3$, which are in the column space OR in the nullspace (or in both) of matrix $\begin{bmatrix}1 & 2 & -3\\1 & 2 & -3\\1 & 2 & -3\end{bmatrix}$.

Solution ....

a) Yes, it is a subspace since the concerned vectors $\boldsymbol{x}$ form a left nullsapce of matrix $\begin{bmatrix}1\\2\\3\end{bmatrix}$.

b) No. An counterexample is that both $(1, 1)$ and $(1, -1)$ satisfy $x^2 = y^2$, but $(1, 1) + (1, -1) = (2, 0)$ does not.

c) No because $(0, 0)$ does not satisfy $x + y = 1$.

d) Yes. Since the column space of $\mathsf{A} = \begin{bmatrix}1 & 2 & -3\\1 & 2 & -3\\1 & 2 & -3\end{bmatrix}$ is spanned by $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and its nullspace is spanned by $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\0\\1\end{bmatrix}$, and since $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}-2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\0\\1\end{bmatrix}$, the column space of A is a subspace of its nullspace, which means that all such $\boldsymbol{x}$ form a subspace as the nullspace of matrix $\begin{bmatrix}1 & 2 & -3\\1 & 2 & -3\\1 & 2 & -3\end{bmatrix}$ is a vector space.

Problem 4 ***Complete Solution***

Consider the matrix

$$\mathsf{A} = \begin{bmatrix}1 & 2 & 1 & 2 & 2\\-1 & -2 & 0 & 0 & -1\\1 & 2 & 0 & 0 & 1\end{bmatrix}$$

a) (8%) Find the complete solution of $\mathsf{A}\boldsymbol{x} = \mathbf{0}$.

b) (8%) Find the complete solution of $\mathsf{A}\boldsymbol{x} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$.

Solution ....

a) We first reduce A as

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ -1 & -2 & 0 & 0 & -1 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & -2 & -1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By setting these free variables as one individually, we get the bases of $\boldsymbol{N}(\mathsf{A})$ which are

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{x} = c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

for any $c_1, c_2, c_3 \in \mathbb{R}$

b) Matrix $\begin{bmatrix} \mathsf{A} & \begin{matrix} 2 \\ 1 \\ -1 \end{matrix} \end{bmatrix}$ can be reduced to

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 2 & 2 \\ -1 & -2 & 0 & 0 & -1 & 1 \\ 1 & 2 & 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By setting all free variables as zero, the particular solution is

$$\boldsymbol{x}^{(\mathrm{p})} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the complete solution is

$$\boldsymbol{x} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

for any $c_1, c_2, c_3 \in \mathbb{R}$

Problem 5 *Rank of Matrix*

Let matrix A be m by n and matrix B be n by m. Suppose that $\mathsf{AB} = \mathsf{I}_{m\times m}$.

a) (6%) Let r denote the rank of the matrix A. What is the general relation between r and m (i.e., $r \geq m$, or $r \leq m$, or $r = m$)? You should justify your answer.

b) (6%) Which one could we conclude, $m \leq n$ or $m \geq n$? Why?

Solution ....

a) Since

$$\text{rank}(\mathsf{A}) \geq \text{rank}(\mathsf{AB}) = \text{rank}(\mathsf{I}) = m$$

and

$$\text{rank}(\mathsf{A}) \leq \min\{m, n\},$$

we have

$$m \leq r \leq \min\{m, n\}. \tag{1}$$

This implies $r = m$.

b) Based on the same reason in (1), we know $n \geq m$.

Problem 6 — *Rank of Matrix*

Let the vector space $\mathbb{V}$ consist of the following eight vectors:

$$\mathbb{V} \triangleq \left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix}, \begin{bmatrix} b \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \\ b \end{bmatrix} \right\},$$

where a, b are two symbols. Define the vector-to-vector addition as component-wise addition through

$$a + a = a, \quad a + b = b, \quad b + a = b, \quad b + b = a.$$

For example,

$$\begin{bmatrix} b \\ a \\ a \end{bmatrix} + \begin{bmatrix} b \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \\ b \end{bmatrix}$$

Define the scaler-to-vector multiplication as:

$$a \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \quad b \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

for any $x_1, x_2, x_3 \in \{a, b\}$. Let $\mathsf{A} = \begin{bmatrix} b & a \\ b & a \\ a & b \end{bmatrix}$. Answer the following questions.

a) (8%) List all the vectors in the column space $\boldsymbol{C}(\mathsf{A})$ of A.

Hint: Linear combinations of the two columns in A.

b) (8%) Determine the null space $\boldsymbol{N}(\mathsf{A})$ of A.

Hint: The nullspace of A consists of all the solutions $\mathsf{A}\boldsymbol{x} = \begin{bmatrix} a \\ a \\ a \end{bmatrix}$.

Solution ....

a) $\boldsymbol{C}(\mathsf{A})$ consists of all the linear combinations of

$$c_1 \begin{bmatrix} b \\ b \\ a \end{bmatrix} + c_2 \begin{bmatrix} a \\ a \\ b \end{bmatrix}$$

for $c_1, c_2 \in \{a, b\}$, which is

$$\left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ b \\ b \end{bmatrix} \right\}.$$

b) In order to solve

$$\begin{bmatrix} b & a \\ b & a \\ a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \cdot \begin{bmatrix} b \\ b \\ a \end{bmatrix} + x_2 \cdot \begin{bmatrix} a \\ a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \end{bmatrix},$$

we take $(x_1, x_2) = (a, a), (a, b), (b, a), (b, b)$ into the above equation and obtain that:

$$\boldsymbol{N}(\mathsf{A}) = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} \right\}.$$