Linear Algebra

Spring Semester 2014
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Midterm Exam 3 of 15 May, 2014

Problem 1 ***Determinant of Orthogonal Matrices***

a) (6%) Prove that every orthonormal matrix Q (i.e., $\mathsf{Q}^{\mathsf{T}}\mathsf{Q} = \mathsf{I}$) has determinant 1 or -1.

b) (6%) Give an orthonormal matrix H as

$$\mathsf{H} \triangleq \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Determine whether $\det(\mathsf{H}) = 1$ or -1.

Hint: One could obtain this determinant through Co-factor formula, Leibniz formula, or Pivot formula.

c) (6%) Find the determinant of another orthonormal matrix H' having the form

$$\mathsf{H}' = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Hint: Relation between H *and* H'.

Problem 2 ***Determinants for Sparse Matrices***

Let a 5 by 5 matrix A be of the form

$$\mathsf{A} \triangleq \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & w \\ 0 & 0 & u & 0 & v \end{bmatrix}.$$

Find the determinant of A by answering the following questions.

a) (8%) How many nonzero product terms are there in the Leibniz formula? List all the nonzero product terms.

b) (8%) Using the Leibniz formula to obtain $\det(\mathsf{A})$.

Hint: ***Leibniz formula:***

$$\det(\mathsf{A}) = \sum_{\sigma \in \mathsf{P}_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where P_n *is the set of all permutation of* $(1, 2, \ldots, n)$*, and*

$$sgn(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ can be recovered to } (1,2,\ldots,n) \text{ by even number} \\ & \text{of pairwise switching} \\ -1 & \text{otherwise} \end{cases}.$$

Problem 3 (16%) ***Determinant***

Consider the following matrices:

$$\mathsf{Z} \triangleq \begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 15 & 26 & 37 & 49 & 0 \end{bmatrix} \quad \text{and} \quad \mathsf{A} \triangleq \begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 15 & 26 & 37 & 49 & 3 \end{bmatrix}.$$

What is $\det(\mathsf{Z})$? What is $\det(\mathsf{A})$?

Hint: Z *is a skew-symmetric matrix (i.e.,* $\mathsf{Z}^\intercal = -\mathsf{Z}$*), and* 3 *is the only non-zero diagonal entry in* A*. You may wish to use the linearity property of determinant, i.e., Property 3, to relate* $\det(\mathsf{Z})$ *and* $\det(\mathsf{A})$*. Please be reminded that performing row eliminations does not change the determinant.*

Problem 4 (14%) ***Cross Product and Triple Product***

Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Prove that $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$.

Problem 5 ***Orthogonality***

Let $\boldsymbol{p}$ be the vector in $\boldsymbol{C}(\mathsf{A}_{m\times n})$ that is nearest to $\boldsymbol{b}$ in the sense that

$$\|\boldsymbol{p} - \boldsymbol{b}\|^2 = \min_{\boldsymbol{v} \in \boldsymbol{C}(\mathsf{A})} \|\boldsymbol{v} - \boldsymbol{b}\|^2,$$

which implies $(\boldsymbol{p} - \boldsymbol{b}) \perp \boldsymbol{p}$. Answer the following questions.

a) (6%) Suppose A has independent columns. Then, is this $\boldsymbol{p}$ unique? In other words, can two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ in $\boldsymbol{C}(\mathsf{A})$ satisfy

$$\|\boldsymbol{p} - \boldsymbol{b}\|^2 = \|\boldsymbol{q} - \boldsymbol{b}\|^2 = \min_{\boldsymbol{v} \in \boldsymbol{C}(\mathsf{A})} \|\boldsymbol{v} - \boldsymbol{b}\|^2 \quad \text{and} \quad \boldsymbol{p} \neq \boldsymbol{q}?$$

If the answer is yes, prove it. If the answer is negative, give a counterexample.

b) (6%) Suppose A has independent columns, and suppose $\hat{\boldsymbol{x}}$ satisfies

$$\|\mathsf{A}\hat{\boldsymbol{x}} - \boldsymbol{b}\|^2 = \min_{\boldsymbol{x} \in \mathbb{R}^n} \|\mathsf{A}\boldsymbol{x} - \boldsymbol{b}\|^2.$$

Is this $\hat{\boldsymbol{x}}$ unique? In other words, can $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ in $\mathbb{R}^n$ satisfy

$$\|\mathsf{A}\hat{\boldsymbol{x}} - \boldsymbol{b}\|^2 = \|\mathsf{A}\hat{\boldsymbol{y}} - \boldsymbol{b}\|^2 = \min_{\boldsymbol{x} \in \mathbb{R}^n} \|\mathsf{A}\boldsymbol{x} - \boldsymbol{b}\|^2 \quad \text{and} \quad \hat{\boldsymbol{x}} \neq \hat{\boldsymbol{y}}?$$

If the answer is yes, prove it. If the answer is negative, give a counterexample.

c) (8%) Answer the previous two subproblems if the columns of A are linearly dependent?

Problem 6 — QR ***Decomposition***

a) (8%) Write $\mathsf{A}_{3\times3}$ as $\mathsf{Q}_{3\times3}\mathsf{R}_{3\times3}$ in terms of QR decomposition, where $\mathsf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

b) (8%) Determine the projection matrix P onto the column space of A.