Linear Algebra

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Midterm Exam 3 of 15 May, 2014

Problem 1 ***Determinant of Orthogonal Matrices***

a) (6%) Prove that every orthonormal matrix Q (i.e., $\mathsf{Q}^{\mathsf{T}}\mathsf{Q} = \mathsf{I}$) has determinant 1 or −1.

b) (6%) Give an orthonormal matrix H as

$$\mathsf{H} \triangleq \frac{1}{2}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Determine whether $\det(\mathsf{H}) = 1$ or -1.

Hint: One could obtain this determinant through Co-factor formula, Leibniz formula, or Pivot formula.

c) (6%) Find the determinant of another orthonormal matrix H' having the form

$$\mathsf{H}' = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Hint: Relation between H *and* H'.

Solution ....

a) From $\det(\mathsf{Q}\mathsf{Q}^{\mathsf{T}}) = \det(\mathsf{Q}) \cdot \det(\mathsf{Q}^{\mathsf{T}}) = (\det(\mathsf{Q}))^2 = \det(\mathsf{I}) = 1$, it follows that $\det(\mathsf{Q}) = \pm 1$.

b) One could obtain this determinant using Co-factor formula, Leibniz formula, or Gauss-Jordan method. We do this by using Gauss-Jordan method and Co-factor formula.

$$\begin{aligned}
\det(\mathsf{H}) &= \left(\frac{1}{2}\right)^4 \det\left(\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}\right) \\
&= \frac{1}{16}\det\left(\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{bmatrix}\right) = \frac{1}{16} \times (1) \times \det\left(\begin{bmatrix} 0 & 2 & 2 \\ 2 & -2 & 0 \\ 2 & 0 & -2 \end{bmatrix}\right) \\
&= \frac{1}{16} \times \left(-2(-4) + 2(0 - (-4))\right) = 1.
\end{aligned}$$

c) Observe that one could obtain H from H' by switching the first and the fourth columns, the second and the third columns, and finally the first and the second columns. Hence,

$$\det(\mathsf{H}') = (-1)^3 \det(\mathsf{H}) = -1.$$

Let a 5 by 5 matrix A be of the form

$$\mathsf{A} \triangleq \begin{bmatrix} a & b & c & d & e \\ f & g & h & i & j \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & w \\ 0 & 0 & u & 0 & v \end{bmatrix}.$$

Find the determinant of A by answering the following questions.

a) (8%) How many nonzero product terms are there in the Leibniz formula? List all the nonzero product terms.

b) (8%) Using the Leibniz formula to obtain $\det(\mathsf{A})$.

Hint: ***Leibniz formula:***

$$\det(\mathsf{A}) = \sum_{\sigma \in \mathsf{P}_n} sgn(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

where P_n *is the set of all permutation of* $(1, 2, \ldots, n)$*, and*

$$sgn(\sigma) = \begin{cases} 1 & \textit{if } \sigma \textit{ can be recovered to } (1,2,\ldots,n) \textit{ by even number} \\ & \textit{of pairwise switching} \\ -1 & \textit{otherwise} \end{cases}.$$

Solution . . . .

a) Consider the important fact about the Leibniz formula: each product term in the Leibniz formula takes exactly one entry from each row/column. In general, there are $5!$ product terms. We firstly focus on the third and fourth rows that have 3 zeros; one can observe that we have only $2!$ nonzero product term choices in the third and fourth rows, i.e., xw and yz. Next, we check the fifth row and found that we can only choose u after picking up xw or yz. For the remaining two rows (i.e., the first and the second rows), $2!$ possibilities are left, i.e., ag and bf. As a result, there are totally $2! \times 2! = 4$ nonzero product terms. One can check that the four non-zero product terms are

$$agxwu, agyzu, bfxwu, bfyzu.$$

b) Apply Leibniz formula to $\det(\mathsf{A})$, we have

$$\begin{aligned} \det(\mathsf{A}) &= \mathrm{sgn}\big(1,2,4,5,3\big)\, agxwu + \mathrm{sgn}\big(1,2,5,4,3\big)\, agyzu \\ &\quad + \mathrm{sgn}\big(2,1,4,5,3\big)\, bfxwu + \mathrm{sgn}\big(2,1,5,4,3\big)\, bfyzu \\ &= agxwu - agyzu - bfxwu + bfyzu, \end{aligned}$$

where $(1,2,4,5,3)$ needs two number of pairwise switching to $(1,2,3,4,5)$, $(1,2,5,4,3)$ needs one number of pairwise switching to $(1,2,3,4,5)$, $(2,1,4,5,3)$ needs three number of pairwise switching to $(1,2,3,4,5)$, and $(2,1,5,4,3)$ needs two number of pairwise switching to $(1,2,3,4,5)$.

Problem 3 (16%) ***Determinant***

Consider the following matrices:

$$\mathsf{Z} \triangleq \begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 15 & 26 & 37 & 49 & 0 \end{bmatrix} \quad \text{and} \quad \mathsf{A} \triangleq \begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 15 & 26 & 37 & 49 & 3 \end{bmatrix}.$$

What is $\det(\mathsf{Z})$? What is $\det(\mathsf{A})$?

Hint: Z is a skew-symmetric matrix (i.e., $\mathsf{Z}^\intercal = -\mathsf{Z}$), and 3 is the only non-zero diagonal entry in A. You may wish to use the linearity property of determinant, i.e., Property 3, to relate $\det(\mathsf{Z})$ and $\det(\mathsf{A})$. Please be reminded that performing row eliminations does not change the determinant.

Solution . . . .

First, $\det(\mathsf{Z}) = \det(\mathsf{Z}^\intercal) = \det(-\mathsf{Z}) = (-1)^5\det(\mathsf{Z}) = -\det(\mathsf{Z})$ implies that $\det(\mathsf{Z}) = 0$.
Secondly, using the linearity property of determinant, we have

$$\begin{aligned} \det(\mathsf{A}) &= \det\left(\begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 15 & 26 & 37 & 49 & 0 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}\right) \\ &= \det(\mathsf{Z}) + \det\left(\begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}\right) \\ &= 0 + \det\left(\begin{bmatrix} 0 & 1 & 3 & -5 & -15 \\ -1 & 0 & 4 & -6 & -26 \\ -3 & -4 & 0 & -2 & -37 \\ 5 & 6 & 2 & 0 & -49 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}\right) \\ &= 3 \times \det\Bigg(\underbrace{\begin{bmatrix} 0 & 1 & 3 & -5 \\ -1 & 0 & 4 & -6 \\ -3 & -4 & 0 & -2 \\ 5 & 6 & 2 & 0 \end{bmatrix}}_{\mathsf{Y}}\Bigg), \qquad (1) \end{aligned}$$

where (1) follows since we can apply the co-factor formula on the fifth row of the matrix in the previous step. Note that Y is also a skew symmetric matrix but its determinant is not zero![1] We then derive

$$\det\left(\begin{bmatrix} 0 & 1 & 3 & -5 \\ -1 & 0 & 4 & -6 \\ -3 & -4 & 0 & -2 \\ 5 & 6 & 2 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 0 & 1 & 3 & -5 \\ -1 & 0 & 4 & -6 \\ 0 & -4 & -12 & 16 \\ 0 & 6 & 22 & -30 \end{bmatrix}\right) (\text{ row eliminations using 2nd row})$$

[1] By following the same derivation to show $\det(\mathsf{Z}) = 0$, we can only obtain $\det(\mathsf{Y}) = \det(\mathsf{Y}^\intercal) = \det(-\mathsf{Y}) = (-1)^4\det(\mathsf{Y}) = \det(\mathsf{Y})$.

$$
= \det\left(\begin{bmatrix} 0 & 1 & 3 & -5 \\ -1 & 0 & 4 & -6 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 4 & 0 \end{bmatrix}\right) \quad (\text{ row eliminations using 1st row})
$$

$$
= -(-1) \times 1 \times (0 - (-16)) = 16 \quad (\text{Co-factor formula on the first column}).
$$

Therefore, we have $\det(\mathsf{A}) = 3 \times 16 = 48$.

Problem 4 (14%) ***Cross Product and Triple Product***

Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. Prove that $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$.

Solution ....

Since the cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$
\begin{aligned}
\boldsymbol{u} \times \boldsymbol{v} &= \det\left(\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right) \\
&= \boldsymbol{i} \cdot \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) - \boldsymbol{j} \cdot \det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) + \boldsymbol{k} \cdot \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) \\ -\det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \end{bmatrix}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} &= \begin{bmatrix} \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) \\ -\det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \end{bmatrix} \cdot \boldsymbol{u} \\
&= u_1 \cdot \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) - u_2 \cdot \det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) + u_3 \cdot \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right) \\
&= 0.
\end{aligned}
$$

Therefore, $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$. That $\boldsymbol{u} \times \boldsymbol{v}$ is perpendicular to $\boldsymbol{v}$ can be similarly proved, and hence we omit it.

Problem 5 ***Orthogonality***

Let $\boldsymbol{p}$ be the vector in $\boldsymbol{C}(\mathsf{A}_{m\times n})$ that is nearest to $\boldsymbol{b}$ in the sense that

$$
\|\boldsymbol{p} - \boldsymbol{b}\|^2 = \min_{\boldsymbol{v} \in \boldsymbol{C}(\mathsf{A})} \|\boldsymbol{v} - \boldsymbol{b}\|^2,
$$

which implies $(\boldsymbol{p} - \boldsymbol{b}) \perp \boldsymbol{p}$. Answer the following questions.

a) (6%) Suppose A has independent columns. Then, is this $\boldsymbol{p}$ unique? In other words, can two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ in $\boldsymbol{C}(\mathsf{A})$ satisfy

$$\|\boldsymbol{p}-\boldsymbol{b}\|^2 = \|\boldsymbol{q}-\boldsymbol{b}\|^2 = \min_{\boldsymbol{v}\in\boldsymbol{C}(\mathsf{A})} \|\boldsymbol{v}-\boldsymbol{b}\|^2 \quad \text{and} \quad \boldsymbol{p}\neq\boldsymbol{q}?$$

If the answer is yes, prove it. If the answer is negative, give a counterexample.

b) (6%) Suppose A has independent columns, and suppose $\hat{\boldsymbol{x}}$ satisfies

$$\|\mathsf{A}\hat{\boldsymbol{x}}-\boldsymbol{b}\|^2 = \min_{\boldsymbol{x}\in\mathbb{R}^n} \|\mathsf{A}\boldsymbol{x}-\boldsymbol{b}\|^2.$$

Is this $\hat{\boldsymbol{x}}$ unique? In other words, can $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$ in $\mathbb{R}^n$ satisfy

$$\|\mathsf{A}\hat{\boldsymbol{x}}-\boldsymbol{b}\|^2 = \|\mathsf{A}\hat{\boldsymbol{y}}-\boldsymbol{b}\|^2 = \min_{\boldsymbol{x}\in\mathbb{R}^n} \|\mathsf{A}\boldsymbol{x}-\boldsymbol{b}\|^2 \quad \text{and} \quad \hat{\boldsymbol{x}}\neq\hat{\boldsymbol{y}}?$$

If the answer is yes, prove it. If the answer is negative, give a counterexample.

c) (8%) Answer the previous two subproblems if the columns of A are linearly dependent?

Solution ....

a) $\boldsymbol{p}$ is unique, i.e., $\boldsymbol{p}=\boldsymbol{q}$ if they are both the minimizer of $\min_{\boldsymbol{v}\in\boldsymbol{C}(\mathsf{A})}\|\boldsymbol{v}-\boldsymbol{b}\|^2$. It can be proved as follows.

- $\boldsymbol{p}\in\boldsymbol{C}(\mathsf{A})$ and $\boldsymbol{q}\in\boldsymbol{C}(\mathsf{A}) \Rightarrow (\boldsymbol{p}-\boldsymbol{q})\in\boldsymbol{C}(\mathsf{A})$ since $\boldsymbol{C}(\mathsf{A})$ is a vector space.
- $(\boldsymbol{p}-\boldsymbol{b})\in\boldsymbol{C}(\mathsf{A})^\perp$ and $(\boldsymbol{q}-\boldsymbol{b})\in\boldsymbol{C}(\mathsf{A})^\perp \Rightarrow (\boldsymbol{p}-\boldsymbol{b})-(\boldsymbol{q}-\boldsymbol{b})=(\boldsymbol{p}-\boldsymbol{q})\in\boldsymbol{C}(\mathsf{A})^\perp$ since $\boldsymbol{C}(\mathsf{A})^\perp$ is also a vector space.
- The only vector that lies in both $\boldsymbol{C}(\mathsf{A})$ and $\boldsymbol{C}(\mathsf{A})^\perp$ is the all-zero vector. Hence, $\boldsymbol{p}=\boldsymbol{q}$.

b) The answer is yes. The proof will follow the proof of a).

Specifically, $\boldsymbol{p}=\mathsf{A}\hat{\boldsymbol{x}}$ is unique from a). Since A has independent columns, these columns form a basis for $\boldsymbol{C}(\mathsf{A})$. By the fact that all vectors in the space can be represented as a unique linear combination of the basis, such $\hat{\boldsymbol{x}}$ is unique.

c) If columns of A are dependent, then the answer to subproblem a) remains positive, while the answer to subproblem b) changes to negative. A counterexample is as follows. Given

$$\mathsf{A}=\begin{bmatrix}1&1\\0&0\end{bmatrix}, \boldsymbol{b}=\begin{bmatrix}1\\1\end{bmatrix}, \boldsymbol{p}=\begin{bmatrix}1\\0\end{bmatrix},$$

$\hat{\boldsymbol{x}}$ can be any vector satisfying $\hat{x}_1+\hat{x}_2=1$.

Problem 6 QR ***Decomposition***

a) (8%) Write $\mathsf{A}_{3\times3}$ as $\mathsf{Q}_{3\times3}\mathsf{R}_{3\times3}$ in terms of QR decomposition, where $\mathsf{A}=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$.

b) (8%) Determine the projection matrix P onto the column space of A.

Solution ....

a) The answer follows from the following derivation.

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{a}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \\
\boldsymbol{y} &= \boldsymbol{a}_2 - \frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a}_2}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}\boldsymbol{x} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} \\
\boldsymbol{z} &= \boldsymbol{a}_3 - \frac{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a}_3}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}\boldsymbol{x} - \frac{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{a}_3}{\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}}\boldsymbol{y} = \begin{bmatrix}0\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{\frac{1}{2}}{\frac{3}{2}}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix};
\end{aligned}
$$

implies

$$
\begin{aligned}
\boldsymbol{q}_1 &= \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix} \\
\boldsymbol{q}_2 &= \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{2}{\sqrt{6}}\begin{bmatrix}\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\end{bmatrix} \\
\boldsymbol{q}_3 &= \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|} = \frac{\sqrt{3}}{2}\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{2}{3}\end{bmatrix} = \begin{bmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}.
\end{aligned}
$$

Therefore, the QR decomposition is

$$
\mathsf{A} = \mathsf{QR} = \begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3\end{bmatrix}\begin{bmatrix}\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_1 & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_2 & \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{a}_3\\ 0 & \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{a}_2 & \boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{a}_3\\ 0 & 0 & \boldsymbol{q}_3^{\mathrm{T}}\boldsymbol{a}_3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ 0 & \frac{3}{\sqrt{6}} & \frac{1}{\sqrt{6}}\\ 0 & 0 & \frac{2}{\sqrt{3}}\end{bmatrix}.
$$

b)

$$
\mathsf{P} = \mathsf{QQ}^{\mathrm{T}} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}
$$