Linear Algebra

Spring Semester 2014
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Problem 1 (10%) ***Linear Combination of Polynomials***

The polynomials can also be regarded as "vectors." Define two vectors $\mathbf{u} = x^3 - 2x^2 - 5x - 3$ and $\mathbf{w} = 3x^3 - 5x^2 - 4x - 9$, show that the vector $\mathbf{w} = 2x^3 - 2x^2 + 12x - 6$ can be expressed as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, while $\mathbf{s} = 3x^3 - 2x^2 + 7x + 8$ can not.

Solution ....

It is not too difficult to find scalars a and b such that

$$\begin{aligned}2x^3 - 2x^2 + 12x - 6 &= a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)\\ &= (a+3b)x^3 + (-2a-5b)x^2 + (-5a-4b)x + (-3a-9b).\end{aligned}$$

This will lead to the following linear equations:

$$\begin{aligned}a + 3b &= 2\\ -2a - 5b &= -2\\ -5a - 4b &= 12\\ -3a - 9b &= -6.\end{aligned}$$

Hence, the only solution is $(a, b) = (-4, 2)$.
Similarly, in the second case, we wish to show that there are no scalars c, d satisfying

$$\begin{aligned}3x^3 - 2x^2 + 7x + 8 &= c(x^3 - 2x^2 - 5x - 3) + d(3x^3 - 5x^2 - 4x - 9)\\ &= (c+3d)x^3 + (-2c-5d)x^2 + (-5c-4d)x + (-3c-9d).\end{aligned}$$

We obtain the linear equations:

$$\begin{aligned}c + 3d &= 3\\ -2c - 5d &= -2\\ -5c - 4d &= 7\\ -3c - 9d &= 8,\end{aligned}$$

which has no solutions.

Problem 2 (30%) ***Parallelogram Law***

A parallelogram is a quadrilateral (四邊形) where the opposite sides are parallel and equal. Use vectors to show that:

a) The midpoints of the two diagonals (對角線) of a parallelogram are identical. In case the four sides of the parallelogram are equal in lengths, then the two diagonals are perpendicular.

b) Given a parallelogram, the sum of the squares of all four sides lengths is equal to the sum of the squares of two diagonals lengths.

Moreover, show that for any quadrilateral, connect the midpoints of its four sides in order, then the midpoints form a parallelogram.

Solution ....

a) Consider a parallelogram $ABCD$ as shown in Fig. 1. Let M, N be the midpoints of the two diagonals $\overline{AC}$ and $\overline{BD}$, respectively. Let $\overrightarrow{AB}$ denote the vector from point A to point B.

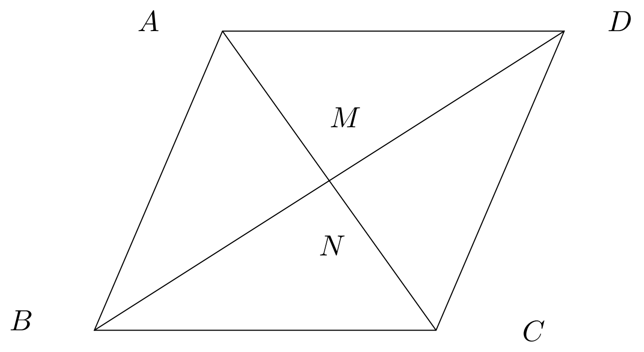

Figure 1: A parallelogram $ABCD$.

Using the method of vectors, we are going to show that

$$\overrightarrow{AM} = \overrightarrow{AN}.$$

Then the two midpoints M, N are identical.

Since M is the midpoint of $\overline{AC}$,

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}).$$

Moreover, that N is the midpoint of $\overline{BD}$ implies

$$\begin{aligned}
\overrightarrow{AN} &= \overrightarrow{AB} + \overrightarrow{BN} \\
&= \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BD} = \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BC} + \overrightarrow{BA}) \qquad (1) \\
&= \overrightarrow{AB} + \frac{1}{2}(\overrightarrow{BC} - \overrightarrow{AB}) \\
&= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) = \overrightarrow{AM},
\end{aligned}$$

where (1) follows directly from the characteristic of the parallelogram.

In case the four sides are identical, we have

$$\begin{aligned}
\overrightarrow{AC} \cdot \overrightarrow{BD} &= (\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} + \overrightarrow{CD}) \\
&= (\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} - \overrightarrow{AB}) \\
&= \|\overrightarrow{BC}\|^2 - \|\overrightarrow{AB}\|^2 = 0.
\end{aligned}$$

I.e., the two diagonals are perpendicular.

b) This property can be shown as follows

$$\begin{aligned}
\|\overrightarrow{AC}\|^2 + \|\overrightarrow{BD}\|^2 &= \|\overrightarrow{AB} + \overrightarrow{BC}\|^2 + \|\overrightarrow{BC} + \overrightarrow{CD}\|^2 \\
&= \|\overrightarrow{AB} + \overrightarrow{BC}\|^2 + \|\overrightarrow{BC} - \overrightarrow{AB}\|^2 \\
&= \|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{BC} + \|\overrightarrow{BC}\|^2 + \|\overrightarrow{AB}\|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{BC} \\
&= 2\left(\|\overrightarrow{AB}\|^2 + \|\overrightarrow{BC}\|^2\right).
\end{aligned}$$

In the second part of the problem, now consider another quadrilateral $ABCD$ as shown in Fig. 2.

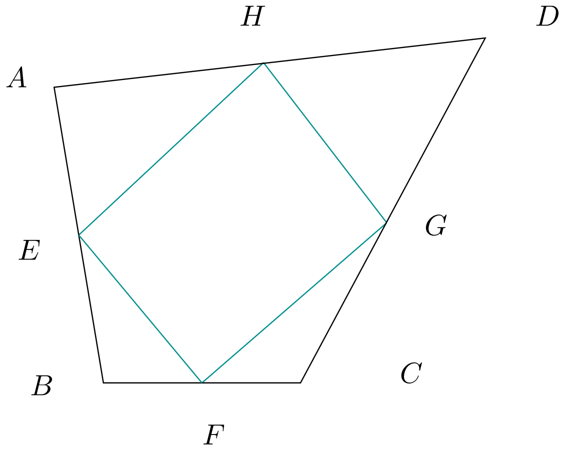

Figure 2: Four midpoints on the four sides in an arbitrary quadrilateral (convex).

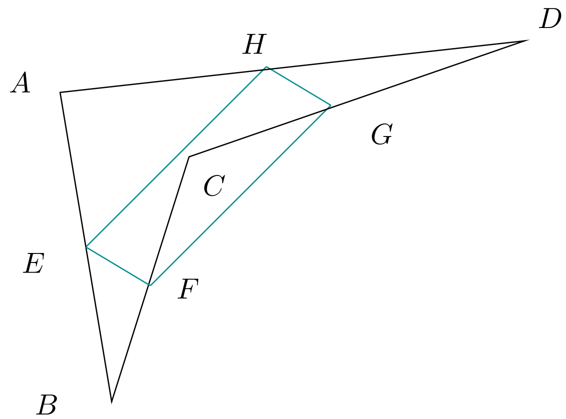

Figure 3: Four midpoints on the four sides in an arbitrary quadrilateral (concave).

Let the midpoints of four sides $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ be E, F, G, H, respectively. In order to prove that the quadrilateral $EFGH$ is a parallelogram, it is sufficient to show that

$$\overrightarrow{EF} = \overrightarrow{HG}.$$

First, observe that vector $\overrightarrow{AC}$ can be written as

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}.$$

Because that the midpoints from E to F can be expressed as

$$\begin{aligned}
\overrightarrow{EF} &= \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{BC}) \\
&= \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{DC}) \\
&= \overrightarrow{HD} + \overrightarrow{DG} = \overrightarrow{HG}.
\end{aligned}$$

This completes the proof.
Note that a quadrilateral like the one in Fig. 2 is called "convex," while the quadrilateral as in Fig 3 is "concave." However, no matter the quadrilateral is convex or concave, the above vector proof holds.

Problem 3 (10%) *Special Angles*

Pick any numbers x, y, z that add to $x + y + z = 0$ and define $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (z, x, y)$. Show that $\mathbf{v} \cdot \mathbf{w} / \|\mathbf{v}\| \|\mathbf{w}\|$ is always $-\frac{1}{2}$. *I.e.*, the angle between $\mathbf{v}$ and $\mathbf{w}$ is $120°$.

Solution . . . .

Let's first pick an example: $\mathbf{v} = (1, 2, -3)$ and $\mathbf{w} = (-3, 1, 2)$. Then we have

$$\cos\vartheta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-3+2-6}{\sqrt{14}\sqrt{14}} = \frac{-7}{14} = -\frac{1}{2}.$$

Hence, $\vartheta = 120°$.
Now let's understand why this is always the case:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= xz + xy + yz \\ &= \frac{1}{2}\left((x+y+z)^2 - x^2 - y^2 - z^2\right) \\ &= \frac{1}{2}\left(0^2 - \sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2}\right) \\ &= -\frac{1}{2}\|\mathbf{v}\|\|\mathbf{w}\|.\end{aligned}$$

Hence,

$$\cos\vartheta = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = -\frac{1}{2}.$$

Problem 4 (10%) *Cauchy–Schwarz Inequality*

For positive real numbers a, b, c. Use Schwarz Inequality to show that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Hint: Consider two vectors $\mathbf{v} = (\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ *and* $\mathbf{w} = (\frac{1}{\sqrt{a+b}}, \frac{1}{\sqrt{b+c}}, \frac{1}{\sqrt{c+a}})$.

Solution . . . .

Based on the Cauchy–Schwarz Inequality, set $\mathbf{v} = (\sqrt{a+b}, \sqrt{b+c}, \sqrt{c+a})$ and $\mathbf{w} = (\frac{1}{\sqrt{a+b}}, \frac{1}{\sqrt{b+c}}, \frac{1}{\sqrt{c+a}})$.
The Schwarz Inequality states that

$$\|\mathbf{v}\|^2\|\mathbf{w}\|^2 \geq \left|\mathbf{v}\cdot\mathbf{w}\right|^2$$

This results in

$$\left((\sqrt{a+b})^2 + \sqrt{b+c})^2 + (\sqrt{c+a})^2\right)\left((\frac{1}{\sqrt{a+b}})^2 + (\frac{1}{\sqrt{b+c}})^2 + (\frac{1}{\sqrt{c+a}})^2\right) \geq (1+1+1)^2 = 3^2,$$

hence, we have

$$(a+b+c)\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) = 1 + \frac{c}{a+b} + 1 + \frac{a}{b+c} + 1 + \frac{b}{c+a} \geq \frac{9}{2}.$$

Problem 5 (10%) *Perpendicular Unit Vectors*

Find four perpendicular unit vectors with all components equal to either $\frac{1}{2}$ or $-\frac{1}{2}$.

Solution . . . .

A unit vector with all components equal to either $\frac{1}{2}$ or $-\frac{1}{2}$ can only be a four-dimensional vector. Check that

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

are the four perpendicular unit vectors we demand.

Problem 6 (10%) — *Linear Combination of Matrices*

Claim that an arbitrary 2×2 matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad a_{ij} \in \mathbb{R}, \forall\, i, j,$$

can be expressed as a linear combination of the four given matrices as below:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

I.e., find four scalars a, b, c, d in terms of a_{11}, a_{12}, a_{21}, and a_{22}, such that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution . . . .

An arbitrary 2×2 matrix can be expressed as a linear combination of the four given matrices as follows:

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \left(\frac{1}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} - \frac{2}{3}a_{22}\right)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &+ \left(\frac{1}{3}a_{11} + \frac{1}{3}a_{12} - \frac{2}{3}a_{21} + \frac{1}{3}a_{22}\right)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \\ &+ \left(\frac{1}{3}a_{11} - \frac{2}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22}\right)\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &+ \left(-\frac{2}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22}\right)\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

Therefore, the four given matrices can *generate* the set of all 2×2 matrices.

Problem 7 (10%) — *Elimination Matrices*

The matrix

$$\mathsf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

needs elimination matrices $\mathsf{E}_{2,1}$, $\mathsf{E}_{3,2}$, and $\mathsf{E}_{4,3}$ to obtain

$$\mathsf{E}_{4,3}\mathsf{E}_{3,2}\mathsf{E}_{2,1}A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix} = \mathsf{U}.$$

a) What are those elimination matrices?

b) Multiply these three elimination matrices to get one matrix E that does the elimination in one step below: $\mathsf{EA} = \mathsf{U}$.

c) What is the inverse of E? *Hint: Think of the inverses of* $\mathsf{E}_{2,1}$, $\mathsf{E}_{3,2}$, *and* $\mathsf{E}_{4,3}$.

Solution ....

a) The first elimination matrix $\mathsf{E}_{2,1}$ can be obtained directly. The determination of the second elimination matrix $\mathsf{E}_{3,2}$ needs to apply the first elimination matrix $\mathsf{E}_{2,1}$ onto A, afterwards the third elimination matrix $\mathsf{E}_{3,2}$ can be resulted.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=\mathsf{E}_{2,1}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix};$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{=\mathsf{E}_{3,2}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix};$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1 \end{bmatrix}}_{=\mathsf{E}_{4,3}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{bmatrix}.$$

b) Multiplication of $\mathsf{E}_{2,1}$, $\mathsf{E}_{3,2}$, and $\mathsf{E}_{4,3}$ yields

$$\mathsf{E}_{4,3}\mathsf{E}_{3,2}\mathsf{E}_{2,1} = \mathsf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{bmatrix}.$$

Note that

$$\begin{aligned} \mathsf{E}_{3,2}\mathsf{E}_{2,1} &= [I + (\mathsf{E}_{3,2} - I)][I + (\mathsf{E}_{2,1} - I)] \\ &= I + (\mathsf{E}_{3,2} - I) + (\mathsf{E}_{2,1} - I) + (\mathsf{E}_{3,2} - I)(\mathsf{E}_{2,1} - I) \\ &\neq I + (\mathsf{E}_{3,2} - I) + (\mathsf{E}_{2,1} - I), \end{aligned}$$

since $(\mathsf{E}_{3,2} - I)(\mathsf{E}_{2,1} - I)$ is not an all-zero matrix.

c) The inverses of $\mathsf{E}_{2,1}$, $\mathsf{E}_{3,2}$, and $\mathsf{E}_{4,3}$ are

$$\mathsf{E}_{2,1}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{E}_{3,2}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathsf{E}_{4,3}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix},$$

which yields

$$\mathsf{E}^{-1} = \mathsf{E}_{2,1}^{-1}\mathsf{E}_{3,2}^{-1}\mathsf{E}_{4,3}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{bmatrix}.$$

Note that

$$\begin{aligned} \mathsf{E}_{2,1}^{-1}\mathsf{E}_{3,2}^{-1} &= [I + (\mathsf{E}_{2,1}^{-1} - I)][I + (\mathsf{E}_{3,2}^{-1} - I)] \\ &= I + (\mathsf{E}_{2,1}^{-1} - I) + (\mathsf{E}_{3,2}^{-1} - I) + (\mathsf{E}_{2,1}^{-1} - I)(\mathsf{E}_{3,2}^{-1} - I) \\ &= I + (\mathsf{E}_{2,1}^{-1} - I) + (\mathsf{E}_{3,2}^{-1} - I), \end{aligned}$$

since $(\mathsf{E}_{2,1}^{-1} - I)(\mathsf{E}_{3,2}^{-1} - I)$ is an all-zero matrix.

Problem 8 (10%) ***Proof of Associativity***

Suppose A is an m by n matrix, B is an n by p matrix, and C is a p by q matrix. Prove $(\mathsf{AB})\mathsf{C} = \mathsf{A}(\mathsf{BC})$ through the following steps:

a) Use the column vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ of B and the components $c_1, \ldots, c_p$ of $\mathbf{c}$ to express $(\mathsf{AB})\mathbf{c}$.

b) Use the column vectors $\mathbf{b}_1, \ldots, \mathbf{b}_p$ of B and the components $c_1, \ldots, c_p$ of $\mathbf{c}$ to express $\mathsf{A}(\mathsf{B}\mathbf{c})$.

c) Explain why these two expressions are identical.

d) Extend $\mathbf{c}$ to matrix $\mathsf{C} = [\mathbf{c}_1\ \mathbf{c}_2\ \cdots\ \mathbf{c}_q]$ and prove $(\mathsf{AB})\mathsf{C} = \mathsf{A}(\mathsf{BC})$.

Solution ....

a) The matrix AB has columns $\mathsf{A}\mathbf{b}_1, \ldots, \mathsf{A}\mathbf{b}_p$ such that $(\mathsf{AB})\mathbf{c}$ equals the linear combination

$$c_1\mathsf{A}\mathbf{b}_1 + \cdots + c_p\mathsf{A}\mathbf{b}_p. \tag{2}$$

b) The vector $\mathsf{B}\mathbf{c}$ is given as $c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p$ such that $\mathsf{A}(\mathsf{B}\mathbf{c})$ equals to

$$\mathsf{A}(c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_p). \tag{3}$$

c) By the distributive law we can write (3) as

$$\mathsf{A}c_1\mathbf{b}_1 + \cdots + \mathsf{A}c_p\mathbf{b}_p.$$

Since $c_1, c_2, \ldots, c_p$ are real numbers and can be taken outside of the multiplication $\mathsf{A}c_i\mathbf{b}_i$, i.e.,

$$\mathsf{A}c_i\mathbf{b}_i = c_i\mathsf{A}\mathbf{b}_i.$$

we see that (2) is equal to (3).

d) If we now consider a general matrix C, we can apply the same argument for each column of $\mathsf{ABC} = [\mathsf{AB}\mathbf{c}_1\ \ \mathsf{AB}\mathbf{c}_2\ \ \cdots\ \ \mathsf{AB}\mathbf{c}_p]$ and prove $(\mathsf{AB})\mathsf{C} = \mathsf{A}(\mathsf{BC})$.