Linear Algebra

Homework 2 of March 06, 2014

Spring Semester 2014
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Problem 1 (30%) ***Properties of Inverses***

Suppose that A is an invertible matrix and has one of the following properties:

a) A is a triangular matrix.

b) A is a symmetric matrix.

c) All the entries of A are integers (i.e. $a_{i,j} \in \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$).

Which of the above would also be true of the inverse of A? Prove your answer if it is true, or find a counterexample if it is false.

Solution ....

a) True.

Please refer to Example 5 in page 86 of textbook, Gilbert Strang, *Introduction to Linear Algebra*, International Fourth Edition, Wellesley Cambridge Press, 2009.

b) True.

Since $\mathsf{A}^{\mathrm{T}} = \mathsf{A}$,

$$\begin{aligned} \mathsf{I} = \mathsf{I}^{\mathrm{T}} &= (\mathsf{A}\mathsf{A}^{-1})^{\mathrm{T}} \\ &= (\mathsf{A}^{-1})^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} \\ &= (\mathsf{A}^{-1})^{\mathrm{T}}\mathsf{A}. \end{aligned}$$

Right-multiply A^{-1} on both sides:

$$\begin{aligned} \mathsf{A}^{-1} &= (\mathsf{A}^{-1})^{\mathrm{T}}\mathsf{A}\mathsf{A}^{-1}. \\ &= (\mathsf{A}^{-1})^{\mathrm{T}}. \end{aligned}$$

We therefore get $\mathsf{A}^{-1} = (\mathsf{A}^{-1})^{\mathrm{T}}$, which indicates A^{-1} is symmetric.

c) False.

A counterexample is

$$\mathsf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

of which the inverse is

$$\mathsf{A}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, one entry of A^{-1} is not an integer.

Problem 2 (20%) *Inverse of Matrix*

Find A^{-1} and B^{-1} by elimination on $[A\ I]$ and $[B\ I]$:

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -1 & 1 & -1 \\ 2 & 2 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 3 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

Solution ....

$$\begin{aligned}
[A \quad I] &= \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 2 & -4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \ldots(1) \\ \ldots(2) \\ \ldots(3) \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} \mathbf{1} & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 6 & -8 & -2 & 0 & 1 \end{array}\right] \begin{array}{l} \ldots(1) \text{ with } \mathbf{1} \text{ the first pivot} \\ \ldots(2)-(-1)\times(1) \\ \ldots(3)-2\times(1) \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & \mathbf{-1} & 1 & 1 & 1 & 0 \\ 0 & 0 & -2 & 4 & 6 & 1 \end{array}\right] \begin{array}{l} \ldots(1) \\ \ldots(2') \text{ with } \mathbf{-1} \text{ the second pivot} \\ \ldots(3')-(-6)\times(2') \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 5 & 6 & 1 \\ 0 & -1 & 0 & 3 & 4 & \frac{1}{2} \\ 0 & 0 & \mathbf{-2} & 4 & 6 & 1 \end{array}\right] \begin{array}{l} \ldots(1)-(-1)\times(3'') \\ \ldots(2')-(-\frac{1}{2})\times(3'') \\ \ldots(3'') \text{ with } \mathbf{-2} \text{ the third pivot} \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & \mathbf{-1} & 0 & 3 & 4 & \frac{1}{2} \\ 0 & 0 & -2 & 4 & 6 & 1 \end{array}\right] \begin{array}{l} \ldots(1')-(2)\times(2'') \\ \ldots(2'') \text{ with } \mathbf{-1} \text{ the fourth pivot} \\ \ldots(3'') \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & -3 & -4 & -\frac{1}{2} \\ 0 & 0 & 1 & -2 & -3 & -\frac{1}{2} \end{array}\right] \begin{array}{l} \ldots(1'') \\ \ldots(2'')\times(-1) \\ \ldots(3'')\times(-\frac{1}{2}) \end{array} \\
&= [I \quad A^{-1}]
\end{aligned}$$

So, $A^{-1} = \begin{bmatrix} -1 & -2 & 0 \\ -3 & -4 & -\frac{1}{2} \\ -2 & -3 & -\frac{1}{2} \end{bmatrix}$.

$$\begin{aligned}
[B \quad I] &= \left[\begin{array}{ccc|ccc} 2 & 3 & 3 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \ldots(1) \\ \ldots(2) \\ \ldots(3) \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} \mathbf{2} & 3 & 3 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 1 \end{array}\right] \begin{array}{l} \ldots(1) \text{ with } \mathbf{2} \text{ the first pivot} \\ \ldots(2)-\frac{1}{2}\times(1) \\ \ldots(3)-\frac{1}{2}\times(1) \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 3 & 1 & 0 & 0 \\ 0 & \mathbf{\frac{1}{2}} & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \ldots(1) \\ \ldots(2') \text{ with } \mathbf{\frac{1}{2}} \text{ the second pivot} \\ \ldots(3')-(1)\times(2') \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 3 & -3 \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \mathbf{1} & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \ldots(1)-(3)\times(3'') \\ \ldots(2')-(-\frac{1}{2})\times(3'') \\ \ldots(3'') \text{ with } \mathbf{1} \text{ the third pivot} \end{array} \\
&\Rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 4 & 0 & -6 \\ 0 & \mathbf{\frac{1}{2}} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \ldots(1')-6\times(2'') \\ \ldots(2'') \text{ with } \mathbf{\frac{1}{2}} \text{ the fourth pivot} \\ \ldots(3'') \end{array}
\end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \begin{array}{l} \ldots (1'') \times \frac{1}{2} \\ \ldots (2'') \times 2 \\ \ldots (3'') \end{array}$$

$$= \begin{bmatrix} \mathsf{I} & \mathsf{B}^{-1} \end{bmatrix}$$

So, $\mathsf{B}^{-1} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$.

Problem 3 (25%) LDU ***Factorization***

Find $\mathsf{A} = \mathsf{LDU}$ factorization for

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 4 \\ 2 & 4 & 6 \end{bmatrix}.$$

(Hint: If $\mathsf{A}^T = \mathsf{A}$*, then* $\mathsf{A} = \mathsf{LDL}^\mathsf{T}$*.)*

Solution . . . .

Since A is symmetric (i.e. $\mathsf{A}^\mathsf{T} = \mathsf{A}$), A can be factorized as

$$\mathsf{A} = \mathsf{LDU} = \mathsf{LDL}^\mathsf{T} = \mathsf{U}^\mathsf{T}\mathsf{DU}.$$

$$\mathsf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 6 & 4 \\ 2 & 4 & 6 \end{bmatrix} \xrightarrow{\text{Forward elimination}} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{DU}.$$

So,

$$\mathsf{A} = \mathsf{U}^\mathsf{T}\mathsf{DU} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{LDU},$$

then

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \mathsf{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \mathsf{U} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 4 (25%) ***Using*** LU ***to Solve*** $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$

Suppose $\mathsf{A} = \mathsf{LU}$ and $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, where L, U, and $\boldsymbol{b}$ are respectively given as follows. Please solve $\boldsymbol{x}$ without knowing/computing A.

$$\mathsf{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{U} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.$$

Solution . . . .

Since $\mathsf{A}\boldsymbol{x} = \mathsf{LU}\boldsymbol{x} = \boldsymbol{b}$. Let $\mathsf{U}\boldsymbol{x} = \boldsymbol{y}$. Solving $\mathsf{L}\boldsymbol{y} = \boldsymbol{b}$ yields:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \boldsymbol{y} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \Rightarrow \boldsymbol{y} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.$$

Then solve $\mathsf{U}\boldsymbol{x} = \boldsymbol{y}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \boldsymbol{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$