# Linear Algebra

## Homework 3 of March 13, 2014

Spring Semester 2014
Prof. Dr. Po-Ning Chen
http://shannon.cm.nctu.edu.tw/la.htm

Deadline: March 20, 2014

### Problem 1 (10%) *Right Inverse of a Non-Square Matrix*

In class, we have illustrated an example of finding the right inverse of a non-square matrix $\mathsf{A}_{2\times 3} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$ (See slides 2-52 to 2-54). Define the 2 by 2 identity matrix as $\mathsf{I}_{2\times 2} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2\end{bmatrix}$. Then the right inverse $\mathsf{R}_{3\times 2}$ of $\mathsf{A}_{2\times 3}$ satisies that $\mathsf{A}_{2\times 3}\mathsf{R}_{3\times 2} = \mathsf{I}_{2\times 2}$. Assume that we have found the elimination matrix $\mathsf{E}_{2\times 2}$ by Gauss-Jordan method such that

$$\mathsf{E}_{2\times 2}\begin{bmatrix}\mathsf{A}_{2\times 3} & \mathsf{I}_{2\times 2}\end{bmatrix} = \begin{bmatrix}\mathsf{E}_{2\times 2}\mathsf{A}_{2\times 3} & \mathsf{E}_{2\times 2}\mathsf{I}\end{bmatrix} = \begin{bmatrix}\mathsf{I}_{2\times 2} & \mathsf{E}_{2\times 2}\mathbf{a}_3 & \mathsf{E}_{2\times 2}\end{bmatrix} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathsf{E}_{2\times 2}\mathbf{a}_3 & \mathsf{E}_{2\times 2}\end{bmatrix}.$$

Using the above representation, show that

$$\mathsf{R}_{3\times 2} = \begin{bmatrix}\mathbf{r}'_1 \\ \mathbf{r}'_2 \\ \mathbf{r}'_3\end{bmatrix} = \begin{bmatrix}\mathsf{E}_{2\times 2} - \mathsf{E}\mathbf{a}_3\mathbf{r}'_3 \\ \mathbf{r}'_3\end{bmatrix},$$

which depends on how we choose the entries of $\mathbf{r}'_3$.

***Solution***

We already know that $\mathsf{EA} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathsf{E}\mathbf{a}_3\end{bmatrix}$. Considering the *outer products* between columns of EA and rows of R, we have

$$\begin{aligned}(\mathsf{EA})\mathsf{R} &= \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathsf{E}\mathbf{a}_3\end{bmatrix}\begin{bmatrix}\mathbf{r}'_1 \\ \mathbf{r}'_2 \\ \mathbf{r}'_3\end{bmatrix} \\ &= \mathbf{e}_1\mathbf{r}'_1 + \mathbf{e}_2\mathbf{r}'_2 + \mathsf{E}\mathbf{a}_3\mathbf{r}'_3 \\ &= \begin{bmatrix}r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2}\end{bmatrix} + \mathsf{E}\mathbf{a}_3\mathbf{r}'_3 \\ &= \mathsf{E}(\mathsf{AR}) = \mathsf{EI} = \mathsf{E}.\end{aligned}$$

This leads to the statement that

$$\begin{bmatrix}r_{1,1} & r_{1,2} \\ r_{2,1} & r_{2,2}\end{bmatrix} = \mathsf{E} - \mathsf{E}\mathbf{a}_3\mathbf{r}'_3.$$

### Problem 2 (10%) *Vandermonde Matrix*

Recall that the *Gauss–Jordan method* can also be used to determine the *determinat*, which is equal to the product of all the pivots before normalization (Slide 2-58). Show that the determinant of the matrix

$$\mathsf{A} \triangleq \begin{bmatrix}1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2\end{bmatrix}$$

is equal to $(b-a)(c-b)(c-a)$ with distinct values $a, b, c \in \mathbb{R}$.

**Solution** ..........................................................................

We only need the pivots of matrix $\mathsf{A}$. Derive

$$\begin{aligned}
&\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} \begin{matrix} \dots(1) \\ \dots(2) \\ \dots(3) \end{matrix} \\
\Rightarrow &\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{bmatrix} \begin{matrix} \dots(1) & \\ \dots(2)-(a)\times(1) & \dots(2') \\ \dots(3)-(a^2)\times(1) & \dots(3') \end{matrix} \\
\Rightarrow &\begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (c-b)(c-a) \end{bmatrix} \begin{matrix} \dots(1) & \\ \dots(2') & \\ \dots(3)-(b+a)\times(2)' & \dots(3'') \end{matrix} \\
\Rightarrow &\begin{bmatrix} 1 & 0 & 1-\frac{c-a}{b-a} \\ 0 & b-a & c-a \\ 0 & 0 & (c-b)(c-a) \end{bmatrix} \begin{matrix} \dots(1)-(\frac{1}{b-a})\times(2') & \dots(1') \\ \dots(2') & \\ \dots(3'') & \end{matrix} \\
\Rightarrow &\begin{bmatrix} 1 & 0 & 0 \\ 0 & b-a & 0 \\ 0 & 0 & (c-b)(c-a) \end{bmatrix} \begin{matrix} \dots(1')-(-\frac{1}{(b-a)(c-a)})\times(3'') & \dots(1'') \\ \dots(2')-(\frac{1}{c-b})\times(3'') & \dots(2'') \\ \dots(3'') & \end{matrix} .
\end{aligned}$$

Hence, the determinant is equal to $(b-a)(c-b)(c-a)$.

## Problem 3 (10%) ***Singular Symmetric Matrix***

Let $k$ be a positive integer and let $\mathsf{I}$ be the $k \times k$ identity matrix. Show that the symmetric matrix

$$\mathsf{A}_{k\times k} \triangleq k\mathsf{I}_{k\times k} - \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \dots & & \\ \vdots & & \ddots & \\ 1 & 1 & \dots & 1 \end{bmatrix}_{k\times k}$$

is *not invertible*.

*Hint: Find a nonzero vector* $\mathbf{x}_{k\times 1}$ *such that* $\mathsf{A}\mathbf{x} = \mathbf{0}$.

**Solution** ..........................................................................

Let $\mathbf{x} = \mathbf{1}_{k\times 1}$ be the all-one vector. Multiply

$$\mathsf{A}\mathbf{1} = \left( k\mathsf{I} - \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & \dots & & \\ \vdots & & \ddots & \\ 1 & 1 & \dots & 1 \end{bmatrix} \right) \mathbf{1} = \begin{bmatrix} k \\ \vdots \\ k \end{bmatrix} - \begin{bmatrix} k \\ \vdots \\ k \end{bmatrix} = \mathbf{0}.$$

Therefore, $\mathsf{A}$ is singular.

## Problem 4 (10%) $\mathsf{LU}$ ***of a Symmetric Matrix***

Compute the $\mathsf{LU}$-factorization of the following symmetric matrix:

$$\mathsf{A} = \begin{bmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{bmatrix}.$$

Find conditions on $a$, $b$, $c$, $d$ such that A has four pivots.

**Solution** ................................................................................................

Do forward elimination in the following steps:

$$\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}\begin{bmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{bmatrix}=\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&b-a&c-a&c-a\\0&b-a&c-a&d-a\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{bmatrix}\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&b-a&c-a&c-a\\0&b-a&c-a&d-a\end{bmatrix}=\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&c-b&d-b\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-1&1\end{bmatrix}\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&c-b&d-b\end{bmatrix}=\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&0&d-c\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}a&a&a&a\\a&b&b&b\\a&b&c&c\\a&b&c&d\end{bmatrix}=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}\begin{bmatrix}a&a&a&a\\0&b-a&b-a&b-a\\0&0&c-b&c-b\\0&0&0&d-c\end{bmatrix}.$$

In order to have four pivots, we need $a \neq 0$, $a \neq b$, $b \neq c$, and $c \neq d$.

**Problem 5 (30%)** ***Some*** $2 \times 2$ ***Symmetric Matrices***

Find $2 \times 2$ symmetric matrices $\mathsf{A} = \mathsf{A}^\intercal$ with these properties:

a) A is not invertible.

b) A is invertible, but cannot be factored into LU (*i.e.*, row exchanges/permutation is necessary).

c) A can be factored into $\mathsf{LDL}^\intercal$, but not into $\mathsf{LL}^\intercal$. (I.e., $\mathsf{A} = \mathsf{LDL}^\intercal$ but $\mathsf{A} \neq \mathsf{LL}^\intercal$.)

**Solution** ................................................................................................

a) Let $\mathsf{A} = \begin{bmatrix}a&c\\c&b\end{bmatrix}$ be symmetric. Then A is singular if the second pivot in

$$\begin{bmatrix}a&c\\0&b-\frac{c^2}{a}\end{bmatrix}$$

is zero, i.e., $ab - c^2 = 0$. For example,

$$\mathsf{A} = \begin{bmatrix}4&6\\6&9\end{bmatrix}.$$

b) We will need a permeation matrix before we factorize the matrix. For example, $\mathsf{A} = \begin{bmatrix}0&a\\a&c\end{bmatrix}$ is symmetric with $a \neq 0$.

c) Let $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ be symmetric. Considering the LU-factorization of A, we have

$$A = \begin{bmatrix} a & c \\ c & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{bmatrix} \begin{bmatrix} a & c \\ 0 & b - \frac{c^2}{a} \end{bmatrix} = LU.$$

It is not too difficult to find some values of $a, b, c$ such that $L \neq U^T$.

For example, $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ gives

$$A = LU = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -4 \end{bmatrix}$$

and therefore can not be written as $LL^T$, but only as

$$A = LDL^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

**Problem 6 (20%)** $PA = LU$ ***Factorization***

Find the $PA = LU$ factorization for

$$\text{a)} \quad A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \qquad \text{b)} \quad A_2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

***Solution*** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

a) We exchange the first two rows before doing the forward elimination:

$$\begin{aligned} P_{1,2}A_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

Hence, we have

$$PA_1 = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

b) Testing with the forward elimination, we obtain:

$$\mathsf{A}_2 = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

So we exchange the last two rows by $\mathsf{P}_{2,3}$, and derive

$$\begin{aligned} \mathsf{P}_{2,3}\mathsf{A}_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{LU}. \end{aligned}$$

**Problem 7 (10%)** ***Inverse of Permutation-like Matrix***

For a permutation matrix

$$\mathsf{P} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

explain why $\mathsf{P}^3 = \mathsf{I}_{3\times 3}$, where $\mathsf{I}$ is the $3 \times 3$ identity matrix.
Next, consider

$$\mathsf{A} = \begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

with $a \neq 0$. Verify that $\mathsf{A}^3 = a\,\mathsf{I}$, and find the inverse of $\mathsf{A}$.

***Solution*** ..........................................................................................

There are only $n! = 3! = 6$ permutations for a sequence of length 3. Hence, among the seven sequences:

$$\mathsf{P}^0\mathbf{x} = \mathbf{x}, \mathsf{P}\mathbf{x}, \mathsf{P}^2\mathbf{x}, \mathsf{P}^3\mathbf{x}, \mathsf{P}^4\mathbf{x}, \mathsf{P}^5\mathbf{x}, \mathsf{P}^6\mathbf{x},$$

there must exist two identical ones. Suppose these two are $\mathsf{P}^i\mathbf{x} = \mathsf{P}^j\mathbf{x}$, where $j > i$. As $\mathbf{x}$ can be arbitrary, we conclude $\mathsf{P}^{j-i} = \mathsf{I}$. This confirms that there must exist an integer $1 \leq k \leq n!$ such that $\mathsf{P}^k = \mathsf{I}$ for any permutation matrix $\mathsf{P}$.
Now let's examine whether the given permutation matrix $\mathsf{P}$ satisfies $\mathsf{P}^3 = \mathsf{I}$. First note that $\mathsf{P} = \mathsf{P}_{1,2}\mathsf{P}_{2,3}$. We then derive

$$\begin{aligned} \mathsf{P}^3 &= \mathsf{P}_{1,2}\mathsf{P}_{2,3}\mathsf{P}_{1,2}\underbrace{\mathsf{P}_{2,3}\mathsf{P}_{1,2}\mathsf{P}_{2,3}}_{=\mathsf{P}_{1,3}} \\ &= \mathsf{P}_{1,2}\underbrace{\mathsf{P}_{2,3}\mathsf{P}_{1,2}\mathsf{P}_{1,3}}_{=\mathsf{P}_{1,2}} \\ &= \mathsf{P}_{1,2}\mathsf{P}_{1,2} = \mathsf{I}. \end{aligned}$$

For the second part, it can be shown that

$$\mathsf{A}(\mathsf{A}^2) = \begin{bmatrix} 0 & 0 & 1 \\ a & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a \\ a & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} = a\,\mathsf{I}.$$

In order to find the inverse of $\mathsf{A}$, we observe

$$\mathsf{A}\left(\frac{1}{a}\mathsf{A}^2\right) = \mathsf{I} = \left(\frac{1}{a}\mathsf{A}^2\right)\mathsf{A}.$$

Accordingly, $\mathsf{A}$ is invertible and $\mathsf{A}^{-1} = \frac{1}{a}\mathsf{A}^2$.