Linear Algebra

Homework 2 of March 27, 2014

Spring Semester 2014
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Problem 1 (20%) ***Column Space v.s. Nullspace***

a) Complete these sentences appropriately for a 3×3 matrix A.

- (2 pts.) If the column space is a (2-dimensional) plane, the nullspace is ________________.
- (2 pts.) If the column space is a (1-dimensional) line, the nullspace is ________________.
- (2 pts.) If the column space is $\mathbb{R}^3$, the nullspace is ________________.
- (2 pts.) If the column space is the zero vector, the nullspace is ________________.

b) (12 pts.) Find a 7×7 matrix A whose column space equals its nullspace, or argue briefly such matrix does not exist.

Solution . . . .

a)
- If the column space is a (2-dimensional) plane, the nullspace is a (1-dimensional) line.
- If the column space is a (1-dimensional) line, the nullspace is a (2-dimensional) plane.
- If the column space is all of $\mathbb{R}^3$, the nullspace is the zero vector.
- If the column space is the zero vector, the nullspace is $\mathbb{R}^3$.

b) (12 pts.) Since "*the dimension of column space*" + "*the dimension of nullspace*" equals to 7, and since the dimension of a space cannot be a fraction such as $\frac{7}{2}$, the column space of A and the nullspace of A cannot be the same.

Problem 2 (30%) ***Applications of Echelon Matrix***

Either construct a matrix A or argue that it is impossible, where the nullspace of A is exactly the multiples of $(2, 3, 4, 1)$ and

a) (10 pts.) A is 2 by 4.

b) (10 pts.) A is 3 by 4.

c) (10 pts.) A is 4 by 4.

Solution . . . .

a) Such matrix A does not exist since a 2-by-4 A has rank at most 2 and the dimension of $\boldsymbol{R}(\mathrm{A})$ + the dimension of $\boldsymbol{N}(\mathrm{A})$ equals 4.

b) Since $\boldsymbol{N}(\mathsf{A})$ consists of all multiples of $\begin{bmatrix}2\\3\\4\\1\end{bmatrix}$, A has 3 pivot variables and one free variable such as

$$\mathsf{A} = \texttt{rref}(\mathsf{A}) = \begin{bmatrix}1 & 0 & 0 & a_1\\0 & 1 & 0 & a_2\\0 & 0 & 1 & a_3\end{bmatrix}.$$

In order to have the nullspace required,

$$\texttt{rref}(\mathsf{A}) = \begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 0 & -3\\0 & 0 & 1 & -4\end{bmatrix}.$$

c) Similar to (b), A contains 3 pivot columns and one free column such as

$$\texttt{rref}(\mathsf{A}) = \begin{bmatrix}1 & 0 & 0 & a_1\\0 & 1 & 0 & a_2\\0 & 0 & 1 & a_3\\0 & 0 & 0 & 0\end{bmatrix}.$$

In order to have the nullspace required,

$$\texttt{rref}(\mathsf{A}) = \begin{bmatrix}1 & 0 & 0 & -2\\0 & 1 & 0 & -3\\0 & 0 & 1 & -4\\0 & 0 & 0 & 0\end{bmatrix}.$$

Problem 3 (20%) ***Special Solutions***

For matrix A below, find the special solutions by setting one of the free variables to 1 and all the other free variables to 0.

$$A = \begin{bmatrix}1 & 1 & -1 & -3\\2 & 0 & -2 & -2\\1 & -1 & -1 & 1\end{bmatrix}$$

Solution . . . .

$\texttt{rref}(\mathsf{A})$ equals

$$\begin{bmatrix}1 & 0 & -1 & -1\\0 & 1 & 0 & -2\\0 & 0 & 0 & 0\end{bmatrix}.$$

Thus, the special solutions are given by

$$s_1 = \begin{bmatrix}a_1\\a_2\\1\\0\end{bmatrix}, \quad s_2 = \begin{bmatrix}b_1\\b_2\\0\\1\end{bmatrix}.$$

Solving $\mathsf{A}s_1 = 0$ and $\mathsf{A}s_2 = 0$, we have the special solutions as:

$$s_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 4 (30%) ***Complete Solutions of*** $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$

Find the basis of $\boldsymbol{C}(\mathsf{A})$ and $\boldsymbol{N}(\mathsf{A})$, and the complete solutions to $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\mathsf{A} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -3 \\ 4 & -1 & 1 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}.$$

Solution ....

$\begin{bmatrix} \mathsf{A} & \boldsymbol{b} \end{bmatrix}$ can be reduced to $\begin{bmatrix} \texttt{rref}(\mathsf{A}) & \boldsymbol{d} \end{bmatrix}$, which is

$$\begin{bmatrix} \texttt{rref}(\mathsf{A}) & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & 2 \\ 0 & 1 & -\frac{7}{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By this, we know that the first and second columns are the pivot columns of A,

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},$$

which forms also the basis of $\boldsymbol{C}(\mathsf{A})$.
Secondly, the special solution of A is

$$\begin{bmatrix} \frac{1}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix},$$

which gives also the basis of $\boldsymbol{N}(\mathsf{A})$.
The complete solutions of $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ can be generally denoted as $\boldsymbol{x}^{(\mathtt{p})} + t\boldsymbol{x}^{(\mathtt{n})}$, where

$$\boldsymbol{x}^{(\mathtt{p})} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}^{(\mathtt{n})} = \begin{bmatrix} \frac{1}{3} \\ \frac{7}{3} \\ 1 \end{bmatrix}.$$