Linear Algebra

Homework 5 of April 03, 2014

Spring Semester 2014
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Problem 1 (20%) ***Bases and Rank***

a) (10%) U comes from A by subtracting row 1 from row 3:

$$\mathsf{A} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \text{ and } \mathsf{U} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Find bases for the two column spaces, $\mathbf{C}(\mathsf{A})$ and $\mathbf{C}(\mathsf{U})$. Find bases for the two row spaces, $\mathbf{R}(\mathsf{A})$ and $\mathbf{R}(\mathsf{U})$. Find bases for the two nullspaces, $\mathbf{N}(\mathsf{A})$ and $\mathbf{N}(\mathsf{U})$.

b) (10%) For which numbers c and d do the below two matrices have exactly rank 2?

$$\mathsf{B} = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad \mathsf{C} = \begin{bmatrix} c & d \\ d & c \end{bmatrix}.$$

Solution . . . .

a) From eliminations, we have

$$\mathsf{E}_{1,3}\mathsf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{U}$$

$$\Rightarrow \mathsf{E}_{1,2}\mathsf{U} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{R},$$

where R is the row reduced echelon form of A. Therefore, based on Slides 3-52 and 3-53, we know $\mathbf{C}(\mathsf{A}) \neq \mathbf{C}(\mathsf{U})$, $\mathbf{R}(\mathsf{A}) = \mathbf{R}(\mathsf{U})$ and $\mathbf{N}(\mathsf{A}) = \mathbf{N}(\mathsf{U})$, and the corresponding pivot columns of R determine the basis of $\mathbf{C}(\mathsf{A})$ and $\mathbf{C}(\mathsf{U})$. We then have the first and second columns of A are the basis for $\mathbf{C}(\mathsf{A})$, the first and second columns of U are the basis for $\mathbf{C}(\mathsf{U})$, and the first and second rows of A or U are bases for $\mathbf{R}(\mathsf{A})$ and $\mathbf{R}(\mathsf{U})$.

To find the nullspaces, consider

$$\mathsf{R} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathsf{I}_{2\times 2} & \mathsf{F}_{2\times 1} \\ \mathsf{O}_{1\times 2} & \mathsf{O}_{1\times 1} \end{bmatrix}.$$

Then

$$\mathsf{N} = \begin{bmatrix} -\mathsf{F}_{2\times 1} \\ \mathsf{I}_{1\times 1} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Hence, the basis for both $\mathbf{N}(\mathsf{A})$ and $\mathbf{N}(\mathsf{U})$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

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b) Since the rank is 2, there are only two pivots. Now if c is the second pivot, the third row must be an all-zero row, which is not possible; hence, $c = 0$. Subtracting the second row from the third row yields

$$\mathsf{B}' = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & d-2 & 0 \end{bmatrix}.$$

This indicates $d = 2$.

On the other hand, C becomes a rank-1 matrix only when the second column is a multiple of the third column and they are both non-zero vectors, i.e.,

$$\begin{bmatrix} c \\ d \end{bmatrix} = k \begin{bmatrix} d \\ c \end{bmatrix} \quad \text{for some } k \in \mathbb{R},$$

and one of c and d is non-zero. The above is valid only when $k = \pm 1$, i.e., $c = \pm d \neq 0$. When $c = d = 0$, C is reduced to rank 0. Thus, any c and d that do not satisfy $c = \pm d$ will make C a rank-2 matrix.

Problem 2 (20%) ***Finding a Basis***

Find a basis for each of the following subspaces of $\mathbb{R}^4$:

a) (5%) All vectors whose components are equal.

b) (5%) All vectors whose components add to zero (also explain why it is a vector space).

c) (5%) All vectors that are perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ (also explain why this forms a vector space).

d) (5%) The column space and the nullspace of $\mathsf{I}_{4\times 4}$.

Solution . . . .

Note that the bases for these vector spaces are not unique!

a) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ for the vector space consisting of all vectors $\begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix}$.

b) The set of all vectors whose components add to zero can be written as the set of vectors $\mathbf{x}$ satisfying

$$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \mathbf{x} = 0.$$

It is a vector space because it is the nullspace of matrix $\mathsf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. The basis can then be obtained from the special solutions:

$$\mathsf{N} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c) Again, the set of vectors that are perpendicular to both $\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ must satisfy

$$\mathsf{A}\mathbf{x} = \begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & 1 & 1\end{bmatrix}\mathbf{x} = \mathbf{0}.$$

So it is the nullspace of A. From the reduced row echelon form of A,

$$\begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & 1 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 1 & 0 & 0\\0 & -1 & 1 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1 & 1\\0 & -1 & 1 & 1\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 0 & 1 & 1\\0 & 1 & -1 & -1\end{bmatrix},$$

the basis of the nullspace of A can be obtained from the two columns of:

$$\mathsf{N} = \begin{bmatrix}-1 & -1\\1 & 1\\1 & 0\\0 & 1\end{bmatrix}.$$

d) Since the rank of identity matrix I is 4, the column space $\mathbf{C}(\mathsf{I})$ is equal to $\mathbb{R}^4$ and the nullspace $\mathbf{N}(\mathsf{I})$ is given by $\{\mathbf{0}\}$. For the former, we choose the standard basis $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$. For the latter, there is no basis; so the set of basis is an empty set.

Problem 3 (20%) ***Basis of a Vector Space of Matrices***

Find a basis and the dimension for each of the spaces formed by the following matrices. Also explain why they are vector spaces.

a) (5%) All diagonal 3×3 matrices.

b) (5%) All symmetric 3×3 matrices ($\mathsf{A}^\mathsf{T} = \mathsf{A}$).

c) (5%) All 3×3 skew-symmetric matrices ($\mathsf{A}^\mathsf{T} = -\mathsf{A}$).

d) (5%) All 2×3 matrices whose columns add to zero.

Solution . . . .

a) It is a vector space because any 3×3 diagonal matrix can be expressed as

$$\begin{bmatrix}d_1 & 0 & 0\\0 & d_2 & 0\\0 & 0 & d_3\end{bmatrix} = d_1\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} + d_2\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} + d_3\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}.$$

The three matrices

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 1\end{bmatrix}$$

are linearly independent because one cannot be represented as a linear combination of the other two. They then form the basis and the dimension of the vector space is 3.

b) Again, it is a vector space since any symmetric matrix can be decomposed to

$$\begin{bmatrix} d_1 & a & b \\ a & d_2 & c \\ b & c & d_3 \end{bmatrix} = d_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + d_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + a \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The basis are the six matrices listed on the right-hand-side above, and the dimension is 6.

c) The reason for the space of all skew-symmetric matrices is a vector space is the same as ??. Note that there are no diagonal entries for skew-symmetric matrices; so the basis should be

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Hence, the dimension is 3.

d) The dimension of this vector space is 3, and the three basis vectors chosen can be

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Problem 4 (20%) ***Extension to Matrix Space***

Determine all the matrices whose nullspace is spanned by $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Hint: Read Slides 3-57 and 3-58.

Solution . . . .

The problem is the same as finding all the matrices whose row space is spanned by the two (linearly independent) vectors perpendicular to $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathsf{F}_{2\times 1} \\ \mathsf{I}_{1\times 1} \end{bmatrix}$. So one choice of the bases of the row space is given by

$$\begin{bmatrix} \mathsf{I}_{2\times 2} & \mathsf{F}_{2\times 1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}.$$

Therefore, all rank-2 $m \times 3$ matrices whose rows are linear combinations of the above two row vectors give the solution (hence, $m \geq 2$), i.e.,

$$\mathsf{A}_{m\times 3} = \mathsf{C}_{m\times 2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}_{2\times 3} = \begin{bmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \\ \vdots & \vdots \\ c_{m,1} & c_{m,2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix},$$

where at least two rows of C are not multiple of each other.

Problem 5 (20%) ***Determination of the Left Nullspace***

a) (10%) Check whether all solutions to matrix equation $\mathsf{A}\mathbf{x} = \mathbf{0}$ are perpendicular to the rows of $\mathsf{R} = \text{rref}(\mathsf{A})$, where

$$\mathsf{A} = \tilde{\mathsf{E}}\mathsf{R} = \begin{bmatrix} 4 & 0 & 0 \\ 8 & 1 & 0 \\ 12 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

b) (10%) Find a basis of the left nullspace of A. Based on Slides 3-60 and 3-61, explain why one choice of bases of the left nullspace of A can be the last row of $\mathsf{E} \triangleq \tilde{\mathsf{E}}^{-1}$.

Solution ....

a) All the solutions of matrix equation $\mathsf{A}\mathbf{x} = \mathbf{0}$ form the nullspace of A; so they are all the linear combinations of the column vectors of:

$$\mathsf{N} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{4} \\ 1 & 0 \\ 0 & -3 \\ 0 & 1 \end{bmatrix}.$$

Since both $\begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{1}{4} \\ 0 \\ -3 \\ 1 \end{bmatrix}$ are perpendicular to the rows of R, so are their linear combinations.

b) Since $\mathsf{A} = \tilde{\mathsf{E}}\mathsf{R}$, we have

$$\tilde{\mathsf{E}}^{-1}\mathsf{A} = \mathsf{E}\mathsf{A} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \mathsf{A} = \begin{bmatrix} \mathsf{E}_{2\times 3} \\ \mathbf{e}_3^{\mathsf{T}} \end{bmatrix} \mathsf{A}_{3\times 4} = \begin{bmatrix} \mathsf{E}_{2\times 3}\mathsf{A}_{3\times 4} \\ \mathbf{e}_3^{\mathsf{T}}\mathsf{A}_{3\times 4} \end{bmatrix} = \mathsf{R} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow \mathbf{e}_3^{\mathsf{T}}\mathsf{A} = \mathbf{0} \Longrightarrow \mathsf{A}^{\mathsf{T}}\mathbf{e}_3 = \mathbf{0}.$$

Therefore, $\mathbf{e}_3 = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}$ that is the last row of E can be a basis of the left nullspace of A.