LINEAR ALGEBRA

Homework 2 of April 17, 2014

Spring Semester 2014
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Deadline: April 24, 2014

Problem 1 (30%) ***Rank of Projection***

P is an n by n projection matrix. Compute the ranks of matrices A, B, and C shown below:

a) (10 pts.) $\mathsf{A} = (\mathsf{I} - \mathsf{P})\mathsf{P}$.

b) (10 pts.) $\mathsf{B} = (\mathsf{I} - \mathsf{P}) - \mathsf{P}$.

c) (10 pts.) $\mathsf{C} = (\mathsf{I} - \mathsf{P})^{1024} + \mathsf{P}^{512}$.

Solution . . . .

a) Since $\mathsf{P}^2 = \mathsf{P}$, $\mathsf{A} = (\mathsf{I} - \mathsf{P})\mathsf{P} = \mathsf{P} - \mathsf{P}^2 = \mathsf{P} - \mathsf{P} = 0$. Therefore, $\text{rank}(\mathsf{A}) = 0$.

b) Since $\mathsf{B} = (\mathsf{I} - \mathsf{P}) - \mathsf{P}$, we have $\mathsf{B}^2 = (\mathsf{I} - \mathsf{P})^2 - (\mathsf{I} - \mathsf{P})\mathsf{P} - \mathsf{P}(\mathsf{I} - \mathsf{P}) + \mathsf{P}^2$. Because

- $(\mathsf{I} - \mathsf{P})^2 = (\mathsf{I} - \mathsf{P})$ due to $(\mathsf{I} - \mathsf{P})$ is also a projection matrix;
- $(\mathsf{I} - \mathsf{P})\mathsf{P} = \mathsf{P}(\mathsf{I} - \mathsf{P}) = 0$ due to (a);
- $\mathsf{P}^2 = \mathsf{P}$,

we obtain $\mathsf{B}^2 = (\mathsf{I} - \mathsf{P})^2 - (\mathsf{I} - \mathsf{P})\mathsf{P} - \mathsf{P}(\mathsf{I} - \mathsf{P}) + \mathsf{P}^2 = (\mathsf{I} - \mathsf{P}) + \mathsf{P} = \mathsf{I}$. Therefore,

$$n \geq \text{rank}(\mathsf{B}) \geq \text{rank}(\mathsf{B}^2) = n.$$

c) Since $\mathsf{C} = (\mathsf{I} - \mathsf{P})^{1024} + \mathsf{P}^{512} = (\mathsf{I} - \mathsf{P}) + \mathsf{P} = \mathsf{I}$, $\text{rank}(\mathsf{C}) = n$.

Problem 2 (30%) ***Projection Matrix***

a) (10 pts.) What matrix P projects every vectors in $\mathbb{R}^3$ onto the line passes through origin and the coordinate point $\boldsymbol{a} = (3, 4, 5)$?

b) (10 pts.) What is the nullspace of matrix P?

c) (10 pts.) What is the row space of matrix P^2?

Solution . . . .

a) Instead of obtaining P directly through $\mathsf{P} = \mathsf{A}(\mathsf{A}^\mathsf{T}\mathsf{A})^{-1}\mathsf{A}$ with $\mathsf{A} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$, we can project each standard basis of $\mathbb{R}^3$ onto the line $\boldsymbol{a}$:

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- The projection of $\boldsymbol{x} = (1,0,0)$ onto $\boldsymbol{a}$ is $\frac{\boldsymbol{a}^T\boldsymbol{x}}{\boldsymbol{a}^T\boldsymbol{a}}\boldsymbol{a} = \begin{bmatrix} 9/50 \\ 12/50 \\ 15/50 \end{bmatrix}$.
- The projection of $\boldsymbol{y} = (0,1,0)$ onto $\boldsymbol{a}$ is $\frac{\boldsymbol{a}^T\boldsymbol{y}}{\boldsymbol{a}^T\boldsymbol{a}}\boldsymbol{a} = \begin{bmatrix} 12/50 \\ 16/50 \\ 20/50 \end{bmatrix}$.
- The projection of $\boldsymbol{z} = (0,0,1)$ onto $\boldsymbol{a}$ is $\frac{\boldsymbol{a}^T\boldsymbol{z}}{\boldsymbol{a}^T\boldsymbol{a}}\boldsymbol{a} = \begin{bmatrix} 15/50 \\ 20/50 \\ 25/50 \end{bmatrix}$.

Therefore, $P = \begin{bmatrix} 9/50 & 12/50 & 15/50 \\ 12/50 & 16/50 & 20/50 \\ 15/50 & 20/50 & 25/50 \end{bmatrix}$.

b) The column space of P is spanned by $\boldsymbol{a}$, thus the rank of $\boldsymbol{N}(\mathsf{P})$ equals 2. Alternatively leting one of free variables be one and the other be zero, we get the bases of $\boldsymbol{N}(\mathsf{P})$ as $\begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$.

Therefore,

$$\boldsymbol{N}(\mathsf{P}) = \left\{ \boldsymbol{x} : \boldsymbol{x} = c_1 \begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix} \text{ for some } c_1, c_2 \in \mathbb{R} \right\}.$$

c) Since $\mathsf{P}^T = \mathsf{P} = \mathsf{P}^2$, the row space of P^2 equals the column space of P, which is spanned by $\boldsymbol{a}$. Therefore, the row space of P^2 is

$$\left\{ \boldsymbol{x} : \boldsymbol{x} = c \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \text{ for some } c \in \mathbb{R} \right\}.$$

Problem 3 (20%) ***The Least Square Solutions***

a) (10 pts.) Suppose $\hat{\boldsymbol{x}}$ is a least square approximation to $\mathsf{A}\boldsymbol{x} = \boldsymbol{b}$ and $\hat{\boldsymbol{y}}$ is a least square approximation to $\mathsf{A}\boldsymbol{y} = \boldsymbol{c}$. Give a least square approximation $\hat{\boldsymbol{z}}$ to $\mathsf{A}\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{c}$?

b) (5 pts.) If Q is an m by n matrix with orthonormal columns, find a least square approximation $\hat{\boldsymbol{x}}$ to $\mathsf{Q}\boldsymbol{x} = \boldsymbol{d}$.

c) (5 pts.) If B = QR, where R is invertible and Q is the same as in (b), find a least square approximation $\hat{\boldsymbol{x}}$ to $\mathsf{B}\boldsymbol{x} = \boldsymbol{e}$.

Solution ....

a) Let P be the projection matrix onto the column space of A. Then $\mathsf{A}\hat{\boldsymbol{x}} = \mathsf{P}\boldsymbol{b}$ and $\mathsf{A}\hat{\boldsymbol{y}} = \mathsf{P}\boldsymbol{c}$, and $\mathsf{A}\hat{\boldsymbol{z}} = \mathsf{P}(\boldsymbol{b} + \boldsymbol{c})$. Thus, a least square approximation to $\mathsf{A}\boldsymbol{z} = \boldsymbol{b} + \boldsymbol{c}$ can be $\hat{\boldsymbol{z}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}$.

b) The least square approximation is $\hat{\boldsymbol{x}} = (\mathsf{Q}^T\mathsf{Q})^{-1}\mathsf{Q}^T\boldsymbol{b}$. Since Q is orthonormal, $\mathsf{Q}^T\mathsf{Q} = \mathsf{I}$. Therefore, $\hat{\boldsymbol{x}} = \mathsf{Q}^T\boldsymbol{d}$.

c) Since $B^TB = R^TQ^TQR = R^TR$, we know B^TB is invertible. Hence, the least square approximation is $\hat{\boldsymbol{x}} = (B^TB)^{-1}B^T\boldsymbol{b} = (R^TQ^TQR)^{-1}R^TQ^T\boldsymbol{b} = (R^TR)^{-1}R^TQ^T\boldsymbol{b} = R^{-1}(R^T)^{-1}R^TQ^T\boldsymbol{b} = R^{-1}Q^T\boldsymbol{b}$.

Problem 4 (20%) QR ***Decomposition***

Write A as QR in terms of QR decomposition, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 6 \end{bmatrix}$.

Solution ....

Derive:

$$
\begin{aligned}
\boldsymbol{x} &= \boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \\
\boldsymbol{y} &= \boldsymbol{a}_2 - \frac{\boldsymbol{x}^T\boldsymbol{a}_2}{\boldsymbol{x}^T\boldsymbol{x}}\boldsymbol{x} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \frac{7}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \\
\boldsymbol{z} &= \boldsymbol{a}_3 - \frac{\boldsymbol{x}^T\boldsymbol{a}_3}{\boldsymbol{x}^T\boldsymbol{x}}\boldsymbol{x} - \frac{\boldsymbol{y}^T\boldsymbol{a}_3}{\boldsymbol{y}^T\boldsymbol{y}}\boldsymbol{y} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} - \frac{29}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \frac{\frac{3}{2}}{\frac{5}{2}}\begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} \frac{29}{14} \\ \frac{29}{7} \\ \frac{87}{14} \end{bmatrix} - \begin{bmatrix} \frac{9}{10} \\ 0 \\ -\frac{3}{10} \end{bmatrix} = \begin{bmatrix} \frac{1}{35} \\ -\frac{1}{7} \\ \frac{3}{35} \end{bmatrix}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\boldsymbol{q}_1 &= \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \\
\boldsymbol{q}_2 &= \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|} = \frac{2}{\sqrt{10}}\begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3/\sqrt{10} \\ 0 \\ -1/\sqrt{10} \end{bmatrix} \\
\boldsymbol{q}_3 &= \frac{\boldsymbol{z}}{\|\boldsymbol{z}\|} = \sqrt{35}\begin{bmatrix} \frac{1}{35} \\ -\frac{1}{7} \\ \frac{3}{35} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{35} \\ -5/\sqrt{35} \\ 3/\sqrt{35} \end{bmatrix}.
\end{aligned}
$$

Therefore, the QR decomposition is

$$
\begin{aligned}
A = QR &= \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^T\boldsymbol{a}_1 & \boldsymbol{q}_1^T\boldsymbol{a}_2 & \boldsymbol{q}_1^T\boldsymbol{a}_3 \\ 0 & \boldsymbol{q}_2^T\boldsymbol{a}_2 & \boldsymbol{q}_2^T\boldsymbol{a}_3 \\ 0 & 0 & \boldsymbol{q}_3^T\boldsymbol{a}_3 \end{bmatrix} \\
&= \begin{bmatrix} 1/\sqrt{14} & 3/\sqrt{10} & 1/\sqrt{35} \\ 2/\sqrt{14} & 0 & -5/\sqrt{35} \\ 3/\sqrt{14} & -1/\sqrt{10} & 3/\sqrt{35} \end{bmatrix} \begin{bmatrix} \sqrt{14} & \sqrt{14}/2 & 29/\sqrt{14} \\ 0 & \sqrt{10}/2 & 3/\sqrt{10} \\ 0 & 0 & 1/\sqrt{35} \end{bmatrix}.
\end{aligned}
$$