LINEAR ALGEBRA

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Spring Semester 2014
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Problem 1 (30%) ***True or False***

True or false? Explain if true, or find a counterexample if false:

a) The determinant of $\mathsf{I}+\mathsf{A}$ is $1+\det(\mathsf{A})$.

b) The determinant of ABC is $|\mathsf{A}||\mathsf{B}||\mathsf{C}|$.

c) The determinant of $4\mathsf{A}$ is $4|\mathsf{A}|$.

d) The determinant of $\mathsf{AB}-\mathsf{BA}$ is zero.

Hint: Try an example with $\mathsf{A}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$.

e) If A is not invertible, then AB is not invertible.

f) The determinant of A is always the product of its pivots.

g) AB and BA have the same determinant.

h) If E is an elementary matrix, then $\det(\mathsf{E})=\pm1$.

i) The determinant of a skew-symmetric matrix ($\mathsf{A}^{\mathsf{T}}=-\mathsf{A}$) is 0.

j) If B is a matrix obtained by interchanging two rows or two columns of A, then $\det(\mathsf{A})=\det(\mathsf{B})$.

Solution ....

a) False! Choose $\mathsf{A}=\mathsf{I}$: $\det(\mathsf{I}+\mathsf{I})=2^n\neq1+\det(\mathsf{I})=2$.

b) True! Apply product rule twice: $\det(\mathsf{ABC})=\det(\mathsf{A})\cdot\det(\mathsf{BC})=\det(\mathsf{A})\cdot\det(\mathsf{B})\cdot\det(\mathsf{C})$.

c) False! The correct rule (stemming from Pivot formula) is $\det(4\mathsf{A})=4^n\det(\mathsf{A})$.

d) False! Choose $\mathsf{A}=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ and $\mathsf{B}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Then $\mathsf{AB}-\mathsf{BA}=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is invertible (and hence its determinant is non-zero).

e) True! If A is not invertible, then $\det(\mathsf{A})=0$. Hence, $\det(\mathsf{AB})=\det(\mathsf{A})\cdot\det(\mathsf{B})=0$, which implies AB is not invertible.

f) False! It could also be $(-\prod\text{pivots})$ if an odd number of row exchanges is required during forward elimination.

g) True! $\det(\mathsf{AB})=\det(\mathsf{A})\cdot\det(\mathsf{B})=\det(\mathsf{B})\cdot\det(\mathsf{A})=\det(\mathsf{BA})$.

h) False! The determinant of an elementary matrix can be larger than 1 (or smaller than -1). It should be easy to find a counterexample.

i) False! Since

$$\det(\mathsf{A}) = \det(\mathsf{A}^\intercal) = \det(-\mathsf{A}) = (-1)^n \det(\mathsf{A}),$$

the statement is true only when n is odd.

j) False! It must be $\det(\mathsf{B}) = -\det(\mathsf{A})$ by Pivot formula.

Problem 2 (10%) ***Reverse Identity Matrices***

The reverse identity matrix J_n is the $n \times n$ matrix

$$\mathsf{J}_n \triangleq \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

E.g.,

$$\mathsf{J}_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Longrightarrow \det(\mathsf{J}_3) = -1$$

$$\mathsf{J}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \det(\mathsf{J}_5) = 1.$$

Find $\det(\mathsf{J}_{301})$ and compute the determinant $\det(\mathsf{J}_n)$ for any positive intger n.

Solution ....

In order to change J_n into the identity matrix, we need to apply $\lfloor \frac{n}{2} \rfloor$ row exchanges: first row with last row, second row with second last row, etc. Hence,

$$\det(\mathsf{J}_n) = (-1)^{\lfloor \frac{n}{2} \rfloor}.$$

Therefore, $\det(\mathsf{J}_{301}) = (-1)^{150} = 1$.

Problem 3 (20%) ***4 by 4 Vandermonde Matrix***

We already know that the determinant of a 3 by 3 Vandermonde matrix

$$\mathsf{V}_3 \triangleq \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

is equal to $(b-a)(c-b)(c-a)$ with distinct values $a, b, c \in \mathbb{R}$. Consider the 4 by 4 Vandermonde matrix

$$\mathsf{V}_4(x) \triangleq \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & x \\ a^2 & b^2 & c^2 & x^2 \\ a^3 & b^3 & c^3 & x^3 \end{bmatrix},$$

where we treat x as an unknown, and a, b, c as constants. Find the determinant of $\mathsf{V}_4(x)$ by the following steps:

a) (10%) Show that the determinant of $\mathsf{V}_4(x)$ can be written as $\det(\mathsf{V}_4(x)) = K(x-a)(x-b)(x-c)$ for some constant K.

b) (10%) Determine the constant K by the Co-factor formula.

Solution ....

a) Since x, x^2, x^3 are all in the same column, and since each product in the Leibniz formula takes exactly one entry from each column, the degree of the polynomial $\det(\mathsf{V}_4(x))$ is at most 3. Taking $x = a, b$ or c into $\det(\mathsf{V}_4(x))$ yields zero determinant since two columns are equivalent. This implies that the three roots of $\det(\mathsf{V}_4) = 0$ are a, b, c. Hence, we can write $\det(\mathsf{V}_4)$ as $K(x-a)(x-b)(x-c)$.

b) To find the constant K, apply the co-factor formula using the fourth column:

$$\begin{aligned}\det(\mathsf{V}_4) &= x^3\det(\mathsf{V}_3) - x^2\mathsf{V}_{3,4} + x\mathsf{V}_{2,4} - \mathsf{V}_{1,4} \\ &= K(x-a)(x-b)(x-c),\end{aligned}$$

where $\mathsf{V}_{i,4}$ are the corresponding co-factors, $i = 1, 2, 3$. Comparing the coefficients of x^3, we know that $K = \det(\mathsf{V}_3) = (b-a)(c-b)(c-a)$.

Problem 4 (20%) ***Fibonacci-Like***

Consider the following sequence of matrices:

$$\mathsf{S}_1 \triangleq \begin{bmatrix}3\end{bmatrix}, \qquad \mathsf{S}_2 \triangleq \begin{bmatrix}3 & 1\\1 & 3\end{bmatrix}, \qquad \mathsf{S}_3 \triangleq \begin{bmatrix}3 & 1 & 0\\1 & 3 & 1\\0 & 1 & 3\end{bmatrix}, \qquad \mathsf{S}_4 \triangleq \begin{bmatrix}3 & 1 & 0 & 0\\1 & 3 & 1 & 0\\0 & 1 & 3 & 1\\0 & 0 & 1 & 3\end{bmatrix}, \qquad \dots$$

a) (5%) Compute $\det(\mathsf{S}_1)$, $\det(\mathsf{S}_2)$, $\det(\mathsf{S}_3)$, and $\det(\mathsf{S}_4)$.

b) (10%) Find a recursive formula for the determinant $\det(\mathsf{S}_n)$.

   Hint: Use the Co-factor formula on the first column.

c) (5%) Prove that the above recursion produces every second Fibonacci number, i.e., prove that $\det(\mathsf{S}_n) = \mathsf{F}_{2n+2}$, where $\mathsf{F}_0 = 0$, $\mathsf{F}_1 = 1$, and $\mathsf{F}_k = \mathsf{F}_{k-1} + \mathsf{F}_{k-2}$.

Solution ....

a) We compute: $\det(\mathsf{S}_1) = 3$, $\det(\mathsf{S}_2) = 8$, $\det(\mathsf{S}_3) = 3\cdot 8 - 1\cdot 3 = 21$, and $\det(\mathsf{S}_4) = 3\cdot 21 - 1\cdot 8 = 55$.

b) From the cofactor rule we see that $\det(\mathsf{S}_n) = 3\det(\mathsf{S}_{n-1}) - \det(\mathsf{S}_{n-2})$.

c) We want to prove that $\det(\mathsf{S}_n) = \mathsf{F}_{2n+2}$. Note that

$$\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3, \dots = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$$

Hence, we see that our claim is correct for $n = 1$ and $n = 2$. Now assume that the claim holds for some $n-1$. We now show that it then also holds for n:

$$\det(\mathsf{S}_n) = 3\det(\mathsf{S}_{n-1}) - \det(\mathsf{S}_{n-2})$$

$$
\begin{aligned}
&= 3\mathsf{F}_{2n} - \mathsf{F}_{2n-2} \\
&= (3\mathsf{F}_{2n-1} + 3\mathsf{F}_{2n-2}) - \mathsf{F}_{2n-2} \\
&= \mathsf{F}_{2n-1} + (2\mathsf{F}_{2n-1} + 2\mathsf{F}_{2n-2}) \\
&= \mathsf{F}_{2n-1} + 2\mathsf{F}_{2n} \\
&= (\mathsf{F}_{2n-1} + \mathsf{F}_{2n}) + \mathsf{F}_{2n} \\
&= \mathsf{F}_{2n+1} + \mathsf{F}_{2n} \\
&= \mathsf{F}_{2n+2}.
\end{aligned}
$$

The claim now follows by induction on n.

Problem 5 (20%) ***Polynomial and Determinant of Matrix***

Let a 4 by 4 matrix A have the form

$$
\mathsf{A} \triangleq \begin{bmatrix} 0 & 0 & 0 & a_0 \\ -1 & 0 & 0 & a_1 \\ 0 & -1 & 0 & a_2 \\ 0 & 0 & -1 & a_3 \end{bmatrix}
$$

Using the steps below to find the determinant of $\det(\mathsf{A} + x\mathsf{I})$, where I is the 4 by 4 identity matrix.

a) (10%) Using the co-factor formula to expand $\det(\mathsf{A} + x\mathsf{I})$ by the first row.

b) (10%) Show that $\det(\mathsf{A} + x\mathsf{I}) = x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$.

Solution ....

a) We have

$$
\mathsf{A} + x\mathsf{I} = \begin{bmatrix} 0 & 0 & 0 & a_0 \\ -1 & 0 & 0 & a_1 \\ 0 & -1 & 0 & a_2 \\ 0 & 0 & -1 & a_3 \end{bmatrix} + \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x & 0 & 0 & a_0 \\ -1 & x & 0 & a_1 \\ 0 & -1 & x & a_2 \\ 0 & 0 & -1 & x + a_3 \end{bmatrix}.
$$

By the co-factor formula, we obtain

$$
\det(\mathsf{A} + x\mathsf{I}) = x \det\left(\begin{bmatrix} x & 0 & a_1 \\ -1 & x & a_2 \\ 0 & -1 & x + a_3 \end{bmatrix}\right) + (-1)^{1+4} a_0 \det\left(\begin{bmatrix} -1 & x & 0 \\ 0 & -1 & x \\ 0 & 0 & -1 \end{bmatrix}\right).
$$

b) From a), directly compute the determinants:

$$
\det\left(\begin{bmatrix} x & 0 & a_1 \\ -1 & x & a_2 \\ 0 & -1 & x + a_3 \end{bmatrix}\right) = x^3 + a_3x^2 + a_2x + a_1;
$$

$$
\det\left(\begin{bmatrix} -1 & x & 0 \\ 0 & -1 & x \\ 0 & 0 & -1 \end{bmatrix}\right) = (-1)^3.
$$

Therefore,

$$
\begin{aligned}
\det(\mathsf{A} + x\mathsf{I}) &= x\left(x^3 + a_3x^2 + a_2x + a_1\right) + (-1)^8 a_0 \\
&= x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.
\end{aligned}
$$

Note that for your information, for an n by n matrix having the form

$$\mathsf{A} + x\mathsf{I} = \begin{bmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ 0 & -1 & x & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & x + a_{n-1} \end{bmatrix},$$

$$\det(\mathsf{A} + x\mathsf{I}) = x^n + \sum_{i=0}^{n-1} a_i x^i.$$