Linear Algebra

Homework 8 of May 1, 2014

Spring Semester 2014
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Problem 1 (30%) ***Triple Product***

Let $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, and $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$.

a) Prove that

$$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} = \det\left(\begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right).$$

b) Compute the volume of the box formed by four vectors $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$.

Solution

a) Since the cross product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\begin{aligned} \boldsymbol{u} \times \boldsymbol{v} &= \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\ &= \boldsymbol{i} \cdot \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) - \boldsymbol{j} \cdot \det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) + \boldsymbol{k} \cdot \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) \\ -\det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} &= \begin{bmatrix} \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) \\ -\det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) \\ \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \end{bmatrix} \cdot \boldsymbol{w} \\ &= w_1 \cdot \det\left(\begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}\right) - w_2 \cdot \det\left(\begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}\right) + w_3 \cdot \det\left(\begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}\right). \end{aligned}$$

b) The volume of this box is the absolute value of the tripe product of vectors $\begin{bmatrix}-2\\2\\-3\end{bmatrix}$, $\begin{bmatrix}-1\\1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\0\\-1\end{bmatrix}$,

i.e.,

$$\begin{aligned}\det\left(\begin{bmatrix}-2 & 2 & -3\\-1 & 1 & 3\\2 & 0 & -1\end{bmatrix}\right) &= 2\cdot\det\left(\begin{bmatrix}2 & -3\\1 & 3\end{bmatrix}\right) - 0\cdot\det\left(\begin{bmatrix}-2 & -3\\-1 & 3\end{bmatrix}\right) - 1\cdot\det\left(\begin{bmatrix}-2 & 2\\-1 & 1\end{bmatrix}\right)\\ &= 2\cdot 9 - 0 - 1\cdot 0 = 18.\end{aligned}$$

Problem 2 (20%) ***Proof of Cramer's Rule***

Let $\boldsymbol{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be the solution of $A\boldsymbol{x} = \boldsymbol{b}$. Prove that

$$x_1 = \frac{\det\left(\begin{bmatrix}b_1 & a_{1,2} & a_{1,3}\\b_2 & a_{2,2} & a_{2,3}\\b_3 & a_{3,2} & a_{3,3}\end{bmatrix}\right)}{\det\left(\begin{bmatrix}a_{1,1} & a_{1,2} & a_{1,3}\\a_{2,1} & a_{2,2} & a_{2,3}\\a_{3,1} & a_{3,2} & a_{3,3}\end{bmatrix}\right)}.$$

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Solution ....

Let X be equal to:

$$X = \begin{bmatrix}x_1 & 0 & 0\\x_2 & 1 & 0\\x_3 & 0 & 1\end{bmatrix}.$$

Then, $\det(X) = x_1$. Since

$$AX = \begin{bmatrix}a_{1,1} & a_{1,2} & a_{1,3}\\a_{2,1} & a_{2,2} & a_{2,3}\\a_{3,1} & a_{3,2} & a_{3,3}\end{bmatrix}\begin{bmatrix}x_1 & 0 & 0\\x_2 & 1 & 0\\x_3 & 0 & 1\end{bmatrix} = \begin{bmatrix}b_1 & a_{1,2} & a_{1,3}\\b_2 & a_{2,2} & a_{2,3}\\b_3 & a_{3,2} & a_{3,3}\end{bmatrix},$$

we obtain

$$\det(AX) = \det(A)\det(X) = \det\begin{bmatrix}b_1 & a_{1,2} & a_{1,3}\\b_2 & a_{2,2} & a_{2,3}\\b_3 & a_{3,2} & a_{3,3}\end{bmatrix}$$

$$\Rightarrow \det(X) = \frac{\det\begin{bmatrix}b_1 & a_{1,2} & a_{1,3}\\b_2 & a_{2,2} & a_{2,3}\\b_3 & a_{3,2} & a_{3,3}\end{bmatrix}}{\det(A)}$$

$$\Rightarrow x_1 = \frac{\det\begin{bmatrix}b_1 & a_{1,2} & a_{1,3}\\b_2 & a_{2,2} & a_{2,3}\\b_3 & a_{3,2} & a_{3,3}\end{bmatrix}}{\det\begin{bmatrix}a_{1,1} & a_{1,2} & a_{1,3}\\a_{2,1} & a_{2,2} & a_{2,3}\\a_{3,1} & a_{3,2} & a_{3,3}\end{bmatrix}}.$$

Problem 3 (30%) ***Cofactor Matrix***

a) Find the cofactor matrix C for

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

b) Compute AC^T.

c) Change $a_{1,3}$ to 40, i.e.,

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 40 \\ 1 & 2 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$$

Then find $\det(\mathsf{A})$.

Solution . . . .

a) $\mathsf{C} = \begin{bmatrix} 6 & -3 & 0 \\ 3 & 1 & -1 \\ -6 & 2 & 1 \end{bmatrix}.$

b) $\mathsf{AC}^\mathsf{T} = \det(\mathsf{A})\mathsf{I} = 3\mathsf{I} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$

c) $\det(\mathsf{A}) = a_{1,1}\mathsf{C}_{1,1} + a_{1,2}\mathsf{C}_{1,2} + a_{1,3}\mathsf{C}_{1,3}$. Since $\mathsf{C}_{1,3} = 0$, and the other four terms, i.e., $a_{1,1}$, $\mathsf{C}_{1,1}$, $a_{1,2}$ and $\mathsf{C}_{1,2}$, remain the same, $\det(\mathsf{A}) = 3$.

Problem 4 (20%) ***Jacobian Matrix***

Let $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, and $z = \rho\cos(\phi)$. Show that the determinant of

$$\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{bmatrix}$$

is equal to $\rho^2\sin(\phi)$.

Solution . . . .

$$\det\left(\begin{bmatrix} \partial x/\partial\rho & \partial x/\partial\phi & \partial x/\partial\theta \\ \partial y/\partial\rho & \partial y/\partial\phi & \partial y/\partial\theta \\ \partial z/\partial\rho & \partial z/\partial\phi & \partial z/\partial\theta \end{bmatrix}\right) = \det\left(\begin{bmatrix} \sin(\phi)\cos(\theta) & \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) & \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \\ \cos(\phi) & -\rho\sin(\phi) & 0 \end{bmatrix}\right)$$
$$= \cos(\phi)\cdot\det\left(\begin{bmatrix} \rho\cos(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) \end{bmatrix}\right)$$

$$
\begin{aligned}
&+\rho\sin(\phi)\cdot\det\left(\begin{bmatrix}\sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta)\\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta)\end{bmatrix}\right)\\
&+0\\
=\ &\cos(\phi)(\rho^2\cos(\phi)\sin(\phi)\cos^2(\theta)+\rho^2\cos(\phi)\sin(\phi)\sin^2(\theta))\\
&+\rho\sin(\phi)(\rho\sin^2(\phi)\cos^2(\theta)+\rho\sin^2(\phi)\sin^2(\theta))\\
=\ &\rho^2\cos^2(\phi)\sin(\phi)(\cos^2(\theta)+\sin^2(\theta))+\rho^2\sin^3(\phi)(\cos^2(\theta)+\sin^2(\theta))\\
=\ &\rho^2\cos^2(\phi)\sin(\phi)+\rho^2\sin^3(\phi)\\
=\ &\rho^2\sin(\phi)(\cos^2(\phi)+\sin^2(\phi))\\
=\ &\rho^2\sin(\phi).
\end{aligned}
$$