Linear Algebra

Homework 9 of May 15, 2014

Spring Semester 2014
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Deadline: May 22, 2014

Problem 1 (20%) ***Eigenvalues***

Consider

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad B = A^2 = \begin{bmatrix} 7 & -3 \\ -2 & 6 \end{bmatrix}.$$

a) (5%) Determine the eigenvalues, λ_1, λ_2 and eigenvectors of A.

b) (5%) Determine the eigenvalues and eigenvectors of B directly based on a).

c) (5%) Why is $\lambda_1^2 + \lambda_2^2 = 13$ in a)?

d) (5%) Show using A how an elementary matrix operation such as row exchange produces different eigenvalues.

Solution . . . .

a) We derive

$$\det\left(\begin{bmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{bmatrix}\right) = -\lambda(-1-\lambda) - 6 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0,$$

and obtain $\lambda_1 = 2$ and $\lambda_2 = -3$. The corresponding eigenvectors are:

$$\lambda_1 = 2\colon \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -3\colon \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Note that eigenvectors can be multiples of the above $\mathbf{x}_1$ and $\mathbf{x}_2$.

b) Eigenvectors $\mathbf{y}_1, \mathbf{y}_2$ are unchanged, and eigenvalues β_1, β_2 are squared:

$$\beta_1 = \lambda_1^2 = 4 \text{ with } \mathbf{y}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\beta_2 = \lambda_2^2 = 9 \text{ with } \mathbf{y}_2 = \mathbf{x}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

This can easily be understood from the Invariance Property 1 of eigenvectors and eigenvalues on class slides: The eigenvectors stay the same for every power of A, and the eigenvalues equal the same power of the respective eigenvalues, *i.e.*,

$$A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda \cdot A\mathbf{x} = \lambda \cdot \lambda\mathbf{x} = \lambda^2\mathbf{x}.$$

c) The trace of $B = A^2$ is equal to 13, and we know that the sum of the eigenvalues is equal to the trace of B.

d) Let's swap the two rows of A and derive its eigenvalues again:

$$\det\left(\begin{bmatrix} 2-\lambda & 0 \\ -1 & 3-\lambda \end{bmatrix}\right) = (2-\lambda)(3-\lambda) = 0.$$

We then obtain $\lambda_1 = 2$ and $\lambda_2 = 3$. So the eigenvalues change due to a simple row exchange.

Problem 2 (20%) ***Eigenvalues of*** AB

Consider

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

a) (10%) Find the eigenvalues and eigenvectors of A, B, AB, and BA.

b) (5%) Are the eigenvalues of AB equal to the eigenvalues of BA?

c) (5%) Are the eigenvalues of AB equal to the eigenvalues of A times the eigenvalues of B?

Solution ....

a) Since A is triangular, we know that $\lambda_1 = 1$ with AM $= 2$. The corresponding eigenvector is:

$$\lambda_1 = 1: \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with GM $= 1$. Similarly, we get for matrix B, $\lambda_1 = 1$ with AM $= 2$. The corresponding eigenvector is:

$$\lambda_1 = 1: \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with GM $= 1$.

For AB, we derive

$$\det\left(\begin{bmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda) - 2 = \lambda^2 - 4\lambda + 1 = 0,$$

and obtain $\lambda_1 = 2 + \sqrt{3}$ and $\lambda_2 = 2 - \sqrt{3}$. The corresponding eigenvectors are

$$\lambda_1 = 2+\sqrt{3}: \begin{bmatrix} -1-\sqrt{3} & 2 \\ 1 & 1-\sqrt{3} \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} -\frac{1}{2} - \frac{\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\lambda_2 = 2-\sqrt{3}: \begin{bmatrix} -1+\sqrt{3} & 2 \\ 1 & 1+\sqrt{3} \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} -\frac{1}{2} + \frac{\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{3}}{2} \end{bmatrix}.$$

For BA, we derive

$$\det\left(\begin{bmatrix} 3-\lambda & 2 \\ 1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda) - 2 = \lambda^2 - 4\lambda + 1 = 0,$$

and obtain $\lambda_1 = 2+\sqrt{3}$ and $\lambda_2 = 2-\sqrt{3}$. The corresponding eigenvectors are:

$$\lambda_1 = 2+\sqrt{3}\colon \begin{bmatrix} 1-\sqrt{3} & 2 \\ 1 & -1-\sqrt{3} \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} \frac{1}{2}-\frac{\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_1 = \begin{bmatrix} 1 \\ -\frac{1}{2}+\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$\lambda_2 = 2-\sqrt{3}\colon \begin{bmatrix} 1+\sqrt{3} & 2 \\ 1 & -1+\sqrt{3} \end{bmatrix} \longrightarrow \text{rref} = \begin{bmatrix} \frac{1}{2}+\frac{\sqrt{3}}{2} & 1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ -\frac{1}{2}-\frac{\sqrt{3}}{2} \end{bmatrix}.$$

b) The eigenvalues of AB and BA are the same. This is always the case if one of A and B is invertible! See the Proposition in Slide 6-28. It is also based on the fact that

$$\begin{aligned} \det(\mathsf{AB}-\lambda\mathsf{I}) &= \det\big(\mathsf{A}^{-1}\mathsf{A}(\mathsf{AB}-\lambda\mathsf{I})\big) \quad (\text{if A is invertible}) \\ &= \det\big(\mathsf{A}^{-1}(\mathsf{AB}-\lambda\mathsf{I})\mathsf{A}\big) = \det(\mathsf{BA}-\mathsf{A}^{-1}\lambda\mathsf{A}) \\ &= \det(\mathsf{BA}-\lambda\mathsf{I}), \end{aligned}$$

which again is related to $\det(\mathsf{AB}) = \det(\mathsf{BA})$ by the product rule. Note that however the eigenvectors of AB and BA are not necessarily the same!

c) The answer is negative! It is true only when A and B share the same eigenvectors. A simple proof is given below.

Suppose $\mathbf{x}$ is an eigenvector for both A and B, respectively corresponding to eigenvalues α and β. Hence we have $\mathsf{AB}\mathbf{x} = \mathsf{A}\beta\mathbf{x} = \beta\mathsf{A}\mathbf{x} = \beta\alpha\mathbf{x}$. In fact, we have learned in class (See the Proposition in Slide 6-38) that this property holds only when $\mathsf{AB} = \mathsf{BA}$, which is not the case here.

Problem 3 (20%) — *Rank, Determinant, and Eigenvalues*

The 3×3 matrix B is known to have eigenvalues 0, 1, and 2. Is this information enough to determine the following (if yes, give the answer and explain; if no, explain why not):

a) (5%) the rank of B;

b) (5%) the determinant of $\mathsf{B}^\mathsf{T}\mathsf{B}$;

c) (5%) the eigenvalues of $\mathsf{B}^\mathsf{T}\mathsf{B}$;

d) (5%) the eigenvalues of $(\mathsf{B}^2+\mathsf{I})^{-1}$.

Solution ....

a) B is diagonalizable since it has 3 different eigenvalues and hence has 3 independent eigenvectors. The two eigenvectors correspond to non-zero eigenvalues span the column space, while the one corresponds to zero eigenvalues span the nullspace. So the rank of B is two.

b) B is singular and so is $\mathsf{B}^\mathsf{T}\mathsf{B}$. Hence $\det(\mathsf{B}^\mathsf{T}\mathsf{B}) = 0$.

c) We only know that 0 is still an eigenvalue of $\mathsf{B}^\mathsf{T}\mathsf{B}$. Apart from this, we cannot say anything. For example, B can be

$$\text{either } \mathsf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad (\text{Both are triangular matrices with eigenvalues } 0, 1, 2.)$$

Then $\mathsf{B}^\mathsf{T}\mathsf{B}$ can be

$$\text{either } \mathsf{B}^\mathsf{T}\mathsf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad (\text{The last eigenvalue of both is apparently different.})$$

d) The eigenvalues of B^2 are 0, 1, 4; the eigenvalues of $\mathsf{B}^2 + \mathsf{I}$ are 1, 2, 5; the eigenvalues of $(\mathsf{B}^2 + \mathsf{I})^{-1}$ are 1, $\frac{1}{2}$, $\frac{1}{5}$ (See Proposition in Slide 6-15.)

Problem 4 (30%) ***Finding Eigenvalues in Different Ways***

Consider the following six matrices:

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\mathsf{C} = \mathsf{A} - \mathsf{I} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \qquad \mathsf{D} = \mathsf{I} - \mathsf{A} = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

$$\mathsf{E} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \qquad \mathsf{F} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}.$$

Find the eigenvalues, rank and determinants of these six matrices.
Note: You do not have to stick to solving $\det(\mathsf{A} - \lambda\mathsf{I}) = 0$. Some quick tricks can be used.

Solution . . . .

a) For A, we see that all rows are identical. So its rank is 1 and its determinant is 0 and at least one of the eigenvalues is 0. Since the dimension of nullspace $\mathbf{N}(\mathsf{A} - 0\mathsf{I})$ is 3, three eigenvalues are equal to 0 (with AM $= 3$). The remaining one should be 4 since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4$ (*i.e.*, four eigenvalues are all 0 is impossible).

b) For B, we see that two rows are independent, and the remaining two are repeated. Hence, $\text{rank}(\mathsf{B}) = 2$, $\det(\mathsf{B}) = 0$, and two eigenvalues must be zero (i.e., GM $= 2$ for zero eigenvalue). To find the remaining two eigenvalues, we make a guess that the two orthogonal rows

$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

are eigenvectors. This guess yields that $\lambda_3 = \lambda_4 = 2$.

c) For C, we can infer from a) that the four eigenvalues are $(0-1, 0-1, 0-1, 4-1) = (-1, -1, -1, 3)$. $\det(\mathsf{C})$ is the product of eigenvalues; so it is equal to $(-1)(-1)(-1)(3) = -3$. Since the determinant is nonzero, C must be invertible; hence, its rank is 4.

d) For D, we get its eigenvalues, determinant and rank similar to c), which are $(1, 1, 1, -3)$, -3, and 4, respectively.

e) For E, since it is an upper triangular matrix, its eigenvalues are the diagonal entries $1, 5, 8, 10$. The determinant is the product of these eigenvalues, *i.e.*, $\det(\mathsf{E}) = 400$. With a non-zero determinant, it has full rank 4.

f) For F, we observe that it is an anti-diagonal matrix. There does not seem to have any trick to handle it but solving $\det(\mathsf{F}-\lambda\mathsf{I})=0$:

$$\begin{aligned}\det(\mathsf{F}-\lambda\mathsf{I}) &= \det\left(\begin{bmatrix}-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 3 & 0 & -\lambda\end{bmatrix}\right)\\ &= \lambda^2(2-\lambda)-1\times 3\times(2-\lambda)\\ &= (2-\lambda)(\lambda^2-3)=0.\end{aligned}$$

Hence, its eigenvalues are $2, \pm\sqrt{3}$. Its determinant is obviously $-6\neq 0$, and therefore its rank is 3.

Problem 5 (10%) ***Power of Matrix***

For $\mathsf{A}=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$, prove that $\mathsf{A}^k=\frac{1}{2}\begin{bmatrix}1+3^k & 1-3^k\\ 1-3^k & 1+3^k\end{bmatrix}$.

Solution . . . .

We derive

$$\det(\mathsf{A}-\lambda\mathsf{I})=(2-\lambda)^2-1=3-4\lambda+\lambda^2=(4-\lambda)(1-\lambda)=0,$$

and obtain $\lambda_1=1$, $\lambda_2=3$. The corresponding eigenvectors are:

$$\begin{aligned}\lambda_1=1\colon &\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\longrightarrow \text{rref}=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} &\implies \mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}\\ \lambda_2=3\colon &\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}\longrightarrow \text{rref}=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} &\implies \mathbf{x}_2=\begin{bmatrix}-1\\ 1\end{bmatrix}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathsf{A}^k &= \left(\mathsf{S}\Lambda\mathsf{S}^{-1}\right)^k=\mathsf{S}\Lambda^k\mathsf{S}^{-1}\\ &= \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1^k & 0\\ 0 & 3^k\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -3^k & 3^k\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1+3^k & 1-3^k\\ 1-3^k & 1+3^k\end{bmatrix}.\end{aligned}$$