Problem 1 (30%) ***Square Root of a Matrix***

Find the matrix $\sqrt{\mathsf{A}} = \mathsf{A}^{1/2}$, where

$$\mathsf{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

(Hint: $\cos\theta + \imath\sin\theta = e^{\imath\theta}$, $\cos\theta - \imath\sin\theta = e^{-\imath\theta}$, and $\imath^2 = -1$. Also, $\mathsf{A} = \mathsf{S}\Lambda\mathsf{S}^{-1}$.)

Solution ....

First, we solve the eigenvalues of A:

$$\begin{aligned}\det(\mathsf{A} - \lambda\mathsf{I}) &= \det\left(\begin{bmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{bmatrix}\right) \\ &= \lambda^2 - 2\lambda\cos\theta + 1 = (\lambda - e^{\imath\theta})(\lambda - e^{-\imath\theta}) = 0\end{aligned}$$

implies that the eigenvalues are $\lambda_1 = e^{\imath\theta}$ and $\lambda_2 = e^{-\imath\theta}$. Then

$$\begin{aligned}\mathsf{A} - \lambda_1\mathsf{I} &= \begin{bmatrix} \cos\theta - e^{i\theta} & -\sin\theta \\ \sin\theta & \cos\theta - e^{i\theta} \end{bmatrix} = \begin{bmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{bmatrix} \\ &\Rightarrow \operatorname{rref}(\mathsf{A} - \lambda_1\mathsf{I}) = \begin{bmatrix} 1 & -\imath \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Therefore, the eigenvector $\boldsymbol{v}_1$ corresponding to eigenvalue λ_1 is

$$\boldsymbol{v}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Similarly, $\mathsf{A} - \lambda_2\mathsf{I}$ can be rref-reduced to

$$\begin{bmatrix} 1 & \imath \\ 0 & 0 \end{bmatrix},$$

and the eigenvector corresponding to eigenvalue λ_2 is

$$\boldsymbol{v}_2 = \begin{bmatrix} -\imath \\ 1 \end{bmatrix}.$$

As a result, $\mathsf{A} = \mathsf{S}\Lambda\mathsf{S}^{-1}$ with

$$\mathsf{S} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}$$

implies

$$\sqrt{\mathsf{A}} = \mathsf{S}\sqrt{\Lambda}\mathsf{S}^{-1} = \mathsf{S}\begin{bmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{bmatrix}\mathsf{S}^{-1} = \begin{bmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{bmatrix}.$$

Problem 2 (25%) ***The Exponential of a Matrix***

Put $\mathsf{A} = \begin{bmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & -2 \end{bmatrix}$ into the infinite series to find $e^{\mathsf{A}t}$.

Solution . . . .

Since

$$\begin{aligned}
\mathsf{A}^2 &= \begin{bmatrix} -1 & -\sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} = -\mathsf{A} \\
\mathsf{A}^3 &= \mathsf{A} \\
&\vdots \\
\mathsf{A}^{2n+1} &= \mathsf{A} \\
\mathsf{A}^{2n+2} &= -\mathsf{A}
\end{aligned}$$

for $n \in \{0, 1, 2, 3, \cdots\}$, we have

$$\begin{aligned}
e^{\mathsf{A}t} &= \mathsf{I} + \mathsf{A}t + \frac{(\mathsf{A}t)^2}{2!} + \frac{(\mathsf{A}t)^3}{3!} + \frac{(\mathsf{A}t)^4}{4!} \cdots \\
&= \mathsf{I} + \mathsf{A}t - \frac{\mathsf{A}t^2}{2!} + \frac{\mathsf{A}t^3}{3!} - \frac{\mathsf{A}t^4}{4!} \cdots \\
&= \mathsf{I} + \left(t - \frac{t^2}{2!} + \frac{t^3}{3!} - \frac{t^4}{4!} \cdots \right) \mathsf{A} \\
&= \mathsf{I} - \left(-t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} \cdots \right) \mathsf{A} \\
&= \mathsf{I} - (e^{-t} - 1)\mathsf{A} \\
&= \mathsf{I} - \begin{bmatrix} (e^{-t} - 1) & \sqrt{2}(e^{-t} - 1) \\ -\sqrt{2}(e^{-t} - 1) & -2(e^{-t} - 1) \end{bmatrix} \\
&= \begin{bmatrix} 1 - (e^{-t} - 1) & -\sqrt{2}(e^{-t} - 1) \\ \sqrt{2}(e^{-t} - 1) & 1 + 2(e^{-t} - 1) \end{bmatrix} \\
&= \begin{bmatrix} 2 - e^{-t} & -\sqrt{2}(e^{-t} - 1) \\ \sqrt{2}(e^{-t} - 1) & 2e^{-t} - 1 \end{bmatrix}.
\end{aligned}$$

Problem 3 (25%) ***Diagonalizing a Matrix***

Diagonalize matrix $\mathsf{A} = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix}$, and find A^{99}.

Solution . . . .

$$\begin{aligned}
\det(\mathsf{A} - \lambda \mathsf{I}) &= \det\left(\begin{bmatrix} 7 - \lambda & 12 \\ -4 & -7 - \lambda \end{bmatrix} \right) \\
&= (7 - \lambda)(-7 - \lambda) + 48 \\
&= -49 + \lambda^2 + 48 \\
&= \lambda^2 - 1 = 0
\end{aligned}$$

implies $\lambda = \pm 1$. The eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and the eigenvector corresponding $\lambda = -1$ is $\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$. This concludes to:

$$\mathsf{A} = \mathsf{S}\Lambda\mathsf{S}^{-1} = \begin{bmatrix} -2 & -\frac{3}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 2 & 4 \end{bmatrix}.$$

Also,

$$\mathsf{A}^{99} = \mathsf{S}\Lambda^{99}\mathsf{S}^{-1} = \mathsf{S}\Lambda\mathsf{S}^{-1} = \mathsf{A} = \begin{bmatrix} 7 & 12 \\ -4 & -7 \end{bmatrix}.$$

Problem 4 (30%) ***Equivalent Conditions***

Let A be an $n \times n$ square matrix. Please show the following two statements are equivalent:

a) The column vectors of A are orthonormal to each other.

b) The row vectors of A are orthonormal to each other.

Solution ....

In order to prove the equivalence of the two statements, the following lemma is required, for which the proof can be found in Slides 2-41∼2-42:

Lemma 1. *For a square matrix, either "both left and right inverses do not exist" or "both left and right inverses exist and equal" is true.*

We now start the proof of the desired equivalence:

a) ⇒ *b)* Let the column vectors of $\mathsf{A} = \begin{bmatrix} \boldsymbol{c}_1 & \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_n \end{bmatrix}$ be orthonormal to each other. Then

$$\mathsf{A}^{\mathsf{T}}\mathsf{A} = \begin{bmatrix} \boldsymbol{c}_1 \cdot \boldsymbol{c}_1 & \boldsymbol{c}_1 \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_1 \cdot \boldsymbol{c}_n \\ \boldsymbol{c}_2 \cdot \boldsymbol{c}_1 & \boldsymbol{c}_2 \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_2 \cdot \boldsymbol{c}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{c}_n \cdot \boldsymbol{c}_1 & \boldsymbol{c}_n \cdot \boldsymbol{c}_2 & \cdots & \boldsymbol{c}_n \cdot \boldsymbol{c}_n \end{bmatrix} = \mathsf{I}$$

Thus, A^{T} is the left inverse of A. By Lemma 1, we know that A^{T} is also the right inverse of A, which implies

$$\mathsf{A}\mathsf{A}^{\mathsf{T}} = \mathsf{I} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{r}_n \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_1 & \boldsymbol{r}_2 & \cdots & \boldsymbol{r}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_1 \cdot \boldsymbol{r}_1 & \boldsymbol{r}_1 \cdot \boldsymbol{r}_2 & \cdots & \boldsymbol{r}_1 \cdot \boldsymbol{r}_n \\ \boldsymbol{r}_2 \cdot \boldsymbol{r}_1 & \boldsymbol{r}_2 \cdot \boldsymbol{r}_2 & \cdots & \boldsymbol{r}_2 \cdot \boldsymbol{r}_n \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{r}_n \cdot \boldsymbol{r}_1 & \boldsymbol{r}_n \cdot \boldsymbol{r}_2 & \cdots & \boldsymbol{r}_n \cdot \boldsymbol{r}_n \end{bmatrix},$$

where $\boldsymbol{r}_i$ denotes the ith row vector of A. Since $\boldsymbol{r}_i \cdot \boldsymbol{r}_j = 1$ if $i = j$, and $\boldsymbol{r}_i \cdot \boldsymbol{r}_j = 0$, otherwise, we conclude that the row vectors of A are orthonormal to each other.

b) ⇒ *a)* A similar proof should work for this part, and hence we omit it.