Linear Algebra

Homework 11 of May 29, 2014

Spring Semester 2014
Prof. Dr. Po-Ning Chen
http://shannon.cm.nctu.edu.tw/la.htm

Deadline: June 05, 2014

Problem 1 (60%) ***True or False***

Answer whether the following statements are true (with reason) or false (with counterexample). Notably, what is required in the answer is not a rigorous proof but simply a reason if the statement is true.

a) Suppose $\mathbf{x}$ is an eigenvector of A with eigenvalue $\lambda = 1$, i.e., $\mathsf{A}\mathbf{x} = \mathbf{x}$. Then $\lambda = -1$ is also an eigenvalue with eigenvector $-\mathbf{x}$.

b) It is possible to find a matrix A that has the diagonal elements being all zero, that is invertible, and that has only real eigenvalues.

c) If the columns of $\mathsf{S}_{n\times n}$ (whose columns are eigenvectors of A) are linearly independent, then

   i) A is invertible.
   ii) A is diagonalizable.
   iii) S is invertible.
   iv) S is diagonalizable.

d) The following statements are about matrix symmetry.

   i) If A has real eigenvalues and eigenvectors, then A is symmetric.
   ii) If $\mathsf{A}_{n\times n}$ has real eigenvalues and n orthogonal eigenvectors, then A is symmetric.
   iii) If an invertible A is symmetric, then A^{-1} is also symmetric.
   iv) If A is symmetric, then its eigen-matrix S is also symmetric.

e) The following statements are about positive-definiteness.

   i) If A is positive definite, then A^{-1} is also positive definite.
   ii) If A is positive definite, then A is invertible.
   iii) A projection matrix (i.e., $\mathsf{P}^2 = \mathsf{P}$) is positive definite.
   iv) A diagonal matrix with positive diagonal entries is positive definite.
   v) A symmetric matrix with a positive determinant is positive definite.

f) The following statements are about matrix similarity.

   i) A symmetric matrix cannot be similar to a non-symmetric matrix.
   ii) An invertible matrix cannot be similar to a singular matrix.
   iii) Matrix A cannot be similar to $-\mathsf{A}$ unless $\mathsf{A} = \mathsf{O}$, where O is the all-zero matrix.
   iv) Matrix A cannot be similar to $\mathsf{A} + \mathsf{I}$.
   v) If A is similar to B, then A^2 is similar to B^2.

Solution ....

a) False because $A(-\mathbf{x}) = (-1)(-\mathbf{x})$ is not necessarily true.

b) True. A example of such is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

c) i) False. If one of the eigenvalues is zero, then the matrix is not invertible. For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

ii) True because we can then write $A = S\Lambda S^{-1}$.

iii) True. Since the square S has independent columns, it is surely invertible.

iv) False. A counterexample is

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = S\Lambda S^{-1},$$

where $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

d) i) False. A counterexample is $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

ii) True. If eigenvectors are orthogonal, then they can be made orthonormal; hence, $A = S\Lambda S^{-1} = S\Lambda S^{\intercal}$, which is symmetric.

iii) True. If A is symmetric, then we can write $A = Q\Lambda Q^{\intercal}$. Hence, its inverse is $A^{-1} = (Q^{\intercal})^{-1}\Lambda^{-1}Q^{-1} = Q\Lambda^{-1}Q^{\intercal}$, which is obviously symmetric.

iv) False. Let us take a counterexample from textbook (Example 1, Section 6.4). The symmetric matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

has eigenvalues 0 and 5 and eigen-matrix

$$S = Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},$$

which is obviously not symmetric.

e) i) True. All eigenvalues of A^{-1} are positive if eigenvalues of A are all positive since eigenvalues of A^{-1} are reciprocals of eigenvalues of A.

ii) True. The determinant is positive because all eigenvalues are positive.

iii) False. Any projection matrix has eigenvalues either 0 or 1 (Proposition in Slide 6-8), so it is not necessary that all eigenvalues are positive. A counterexample is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

iv) True. The eigenvalues of a diagonal matrix are its diagonal entries.

v) False. A counterexample is $-I_{n\times n}$ with n even, which has positive determinant but does not have all positive eigenvalues (all are -1).

f) i) False. For a diagonalizable matrix A, we have

$$A = S\Lambda S^{-1}.$$

So a non-symmetric A can be similar to a symmetric Λ. A counterexample can be easily made.

ii) True. Similar matrices have the same eigenvalues. Hence, it is not possible that an invertible matrix has identical eigenvalues to a singular matrix, which has at least one zero eigenvalue.

iii) False. Consider

$$\mathsf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad -\mathsf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathsf{A} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

So A and $-\mathsf{A}$ are similar but $\mathsf{A} \neq \mathsf{O}$.

iv) True. The eigenvalues of $\mathsf{A} + \mathsf{I}$ are given by those of A by adding one; so the two cannot have the same set of eigenvalues.

v) True. If $\mathsf{A} = \mathsf{M}^{-1}\mathsf{B}\mathsf{M}$, then $\mathsf{A}^2 = \mathsf{M}^{-1}\mathsf{B}\mathsf{M}\mathsf{M}^{-1}\mathsf{B}\mathsf{M} = \mathsf{M}^{-1}\mathsf{B}^2\mathsf{M}$.

Problem 2 (10%) ***Eigenvectors of Symmetric Matrices are Perpendicular***

Here, we provide an alternative proof for the fact that eigenvectors are perpendicular to each other when A is symmetric.

a) (5%) Suppose that $\mathsf{A}\mathbf{x} = \lambda\mathbf{x}$ and $\mathsf{A}\mathbf{y} = 0\mathbf{y}$, where $\lambda \neq 0$. Argue that $\mathbf{x}$ is in the column space of A and that $\mathbf{y}$ is in the nullspace of A. Why are $\mathbf{x}$ and $\mathbf{y}$ perpendicular?

b) (5%) If $\mathsf{A}\mathbf{z} = \beta\mathbf{z}$ for some $\beta \neq \lambda$, apply the argument in a) to $(\mathsf{A} - \beta\mathsf{I})\mathbf{x} = (\lambda - \beta)\mathbf{x}$ and $(\mathsf{A} - \beta\mathsf{I})\mathbf{z} = 0\mathbf{z}$. Show that the eigenvectors with respect to distinct eigenvalues are perpendicular.

Solution ....

a) Any vector of the form $\mathsf{A}\mathbf{x}$ is in the column space of A, so $\mathbf{x}$ is in the column space of A. Since $\mathsf{A}\mathbf{y} = \mathbf{0}$, $\mathbf{y}$ is by definition in the nullspace of A. For a symmetric matrix, its column space and nullspace are orthogonal complement because column space is identical to row space. In summary, we conclude that $\mathbf{x} \cdot \mathbf{y} = 0$.

b) Let $\mathsf{B} \triangleq \mathsf{A} - \beta\mathsf{I}$. Then, $\mathbf{x}$ is in the column space of B and $\mathbf{z}$ is in the nullspace of B. Since B is also symmetric, $\mathbf{x}$ and $\mathbf{z}$ are perpendicular by the statement in a).

Problem 3 (20%) ***Test for Positive Definiteness***

a) (10%) Which of the following matrices only has positive eigenvalues? Do not compute the eigenvalues but use the test for positive definiteness. For those matrices that are not positive definite, find a vector $\mathbf{x}$ such that $\mathbf{x}^\mathsf{T}\mathsf{A}\mathbf{x} \leq 0$.

$$\mathsf{A}_1 = \begin{bmatrix} 5 & 6 \\ 6 & 7 \end{bmatrix}, \quad \mathsf{A}_2 = \begin{bmatrix} -1 & -2 \\ -2 & -5 \end{bmatrix}, \quad \mathsf{A}_3 = \begin{bmatrix} 1 & 10 \\ 10 & 100 \end{bmatrix}, \quad \mathsf{A}_4 = \begin{bmatrix} 1 & 10 \\ 10 & 101 \end{bmatrix}.$$

b) (10%) For what numbers of b and c are the below two matrices positive definite?

$$\mathsf{A} = \begin{bmatrix} 1 & b \\ b & 9 \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} 2 & 4 \\ 4 & c \end{bmatrix}.$$

Factor each matrix into $\mathsf{L}\mathsf{D}\mathsf{L}^\mathsf{T}$.

Solution ....

a) Recall the test for positive definiteness: The determinants of all upper left corner submatrices are positive. In the case of a 2 by 2 matrix, this means that the top left entry must be positive and the determinant must be positive.

A_1: $5 > 0$ but $\det \mathsf{A}_1 = -1 < 0 \Longrightarrow$ not positive definite. We then see that $\mathbf{x}^\intercal \mathsf{A}_1 \mathbf{x} = 5x_1^2 + 12x_1x_2 + 7x_2^2 = (5x_1 + 7x_2)(x_1 + x_2)$ can be negative, *e.g.*, if $x_1 = 5$ and $x_2 = -4$, then $\mathbf{x}^\intercal \mathsf{A}_1 \mathbf{x} = -3$.

A_2: $-1 < 0 \Longrightarrow$ not positive definite. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\mathbf{x}^\intercal \mathsf{A}_2 \mathbf{x} = -10 < 0$.

A_3: $1 > 0$ but $\det \mathsf{A}_3 = 0 \Longrightarrow$ not positive definite. Choose a vector $\mathbf{x} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$ belonging to the nullspace of A_3. Then $\mathbf{x}^\intercal \mathsf{A}_3 \mathbf{x} = 0$.

A_4: $1 > 0$ and $\det \mathsf{A}_4 = 1 \Longrightarrow$ positive definite.

b) We again use the test for positive definiteness:

- $a_{1,1} = 1 > 0$; $\det(\mathsf{A}) = 9 - b^2 > 0 \Longrightarrow -3 < b < 3$.
- $b_{1,1} = 2 > 0$; $\det \mathsf{B} = 2c - 16 > 0 \Longrightarrow c > 8$.

Using forward and backward eliminations, we obtain that

$$\mathsf{A} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 9 - b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 - b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix},$$

and

$$\mathsf{B} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & c - 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & c - 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

Problem 4 (10%) ***Positive Definiteness***

Show that if the columns of $\mathsf{A}_{n\times n}$ are linearly independent and $\mathsf{C}_{n\times n}$ is positive definite, then $\mathsf{A}^\intercal \mathsf{C} \mathsf{A}$ is also positive definite.
Hint: $\mathbf{x}^\intercal \mathsf{A}^\intercal \mathsf{C} \mathsf{A} \mathbf{x}$.

Solution ....

We first note that $\mathbf{x} = 0$ if, and only if $\mathsf{A}\mathbf{x} = \mathbf{0}$. We then derive

$$\mathbf{x}^\intercal \mathsf{A}^\intercal \mathsf{C} \mathsf{A} \mathbf{x} = (\mathsf{A}\mathbf{x})^\intercal \mathsf{C} (\mathsf{A}\mathbf{x}) = \mathbf{y}^\intercal \mathsf{C} \mathbf{y} > 0,$$

where $\mathbf{y} = \mathsf{A}\mathbf{x} \neq \mathbf{0}$ for any non-zero $\mathbf{x}$.