Linear Algebra

Spring Semester 2014
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Problem 1 (30%) ***Singular Value Decomposition***

Answer the following questions for matrix $\mathsf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

a) (10%) Find its eigenvalues and eigenvectors.

b) (10%) Find its Jordan form.

c) (10%) Find its singular value decomposition (SVD).

Solution ......

a)

$$\det(\mathsf{A} - \lambda \mathsf{I}) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 3 & 6-\lambda \end{bmatrix}\right) = (1-\lambda)(6-\lambda) - 6 = \lambda^2 - 7\lambda = 0$$

Thus, the eigenvalues are 0 and 7. For $\lambda = 0$, we derive

$$\text{rref}(\mathsf{A} - 0\mathsf{I}) = \text{rref}(\mathsf{A}) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix},$$

which implies its corresponding eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Similarly for $\lambda = 7$,

$$\text{rref}(\mathsf{A} - 7\mathsf{I}) = \text{rref}\left(\begin{bmatrix} -6 & 2 \\ 3 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix},$$

which implies its corresponding eigenvector is $\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$.

b) From (a), we know $\mathsf{J}_1 = \begin{bmatrix} 0 \end{bmatrix}$, $\mathsf{J}_2 = \begin{bmatrix} 7 \end{bmatrix}$, and

$$\mathsf{J} = \begin{bmatrix} \mathsf{J}_1 & 0 \\ 0 & \mathsf{J}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix}.$$

Then the matrix M can be obtained from the eigenvectors, which is

$$\mathsf{M} = \begin{bmatrix} M_1 & M_2 \end{bmatrix} = \begin{bmatrix} -2 & 1/3 \\ 1 & 1 \end{bmatrix},$$

and $\mathsf{A} = \mathsf{M}\mathsf{J}\mathsf{M}^{-1}$.

c) Denote $\mathsf{A} = \mathsf{U}\Sigma\mathsf{V}^{\mathsf{T}}$, $\mathsf{V}^{-1} = \mathsf{V}^{\mathsf{T}}$ and $\mathsf{U}^{-1} = \mathsf{U}^{\mathsf{T}}$. Then

$$\mathsf{A}\mathsf{A}^{\mathsf{T}} = \mathsf{U}\Sigma\mathsf{V}^{\mathsf{T}}\mathsf{V}\Sigma^{\mathsf{T}}\mathsf{U}^{\mathsf{T}} = \mathsf{U}\Sigma^2\mathsf{U}^{\mathsf{T}} = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

So, U is the matrix of orthonormal eigenvectors of $\mathsf{A}\mathsf{A}^{\mathsf{T}}$, which is

$$\mathsf{U} = \begin{bmatrix} \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}.$$

Similarly,

$$\mathsf{A}^{\mathsf{T}}\mathsf{A} = \mathsf{V}\Sigma^{\mathsf{T}}\mathsf{U}^{\mathsf{T}}\mathsf{U}\Sigma\mathsf{V}^{\mathsf{Y}} = \mathsf{V}\Sigma^2\mathsf{V}^{\mathsf{T}} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}.$$

So, V is the matrix of orthonormal eigenvectors of $\mathsf{A}^{\mathsf{T}}\mathsf{A}$, which is

$$\mathsf{V} = \begin{bmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Σ^2 is the diagonal matrix of eigenvalues of $\mathsf{A}\mathsf{A}^{\mathsf{T}}$ (or $\mathsf{A}^{\mathsf{T}}\mathsf{A}$), which is $\begin{bmatrix} 0 & 0 \\ 0 & 50 \end{bmatrix}$. Therefore

$$\Sigma = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{50} \end{bmatrix}.$$

Problem 2 (20%) ***Properties of Similar Matrices***

True or false? Explain if true, or find a counterexample if false:

a) (5%) An $n \times n$ matrix A is similar to A itself.

b) (5%) If A is similar to B and B is similar to C, then A is similar to C.

c) (5%) If A is similar to B, then they have the same eigenvalues.

d) (5%) If A^2 is similar to B^2, then A is similar to B.

Solution ....

a) True. It is apparent that $\mathsf{A} = \mathsf{M}^{-1}\mathsf{A}\mathsf{M}$, where $\mathsf{M} = \mathsf{I}$.

b) True. Let $\mathsf{A} = \mathsf{M}^{-1}\mathsf{B}\mathsf{M}$ and $\mathsf{B} = \mathsf{N}^{-1}\mathsf{C}\mathsf{N}$. Then $\mathsf{A} = \mathsf{M}^{-1}(\mathsf{N}^{-1}\mathsf{C}\mathsf{N})\mathsf{M} = (\mathsf{N}\mathsf{M})^{-1}\mathsf{C}(\mathsf{N}\mathsf{M}) = \mathsf{K}^{-1}\mathsf{C}\mathsf{K}$ where $\mathsf{K} = \mathsf{N}\mathsf{M}$. Thus, A is similar to C.

c) True. Since $\mathsf{A} = \mathsf{M}^{-1}\mathsf{B}\mathsf{M}$ and $\mathsf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, we infer that $\mathsf{M}^{-1}\mathsf{B}\mathsf{M}\boldsymbol{x} = \lambda\boldsymbol{x}$ implies $\mathsf{B}(\mathsf{M}\boldsymbol{x}) = \lambda(\mathsf{M}\boldsymbol{x})$. Therefore, $\mathsf{M}\boldsymbol{x}$ is the eigenvector of B corresponding to eigenvalue λ.

d) False. A counterexample is as follows. Let

$$\mathsf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathsf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then

$$\mathsf{A}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathsf{B}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Accordingly, A^2 is similar to B^2, but A is not similar to B.

Problem 3 (20%) ***Singular Values***

Suppose $\mathsf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

a) (10%) Find all singular values of A.

b) (10%) Determine $e^{t\mathsf{A}}$.

   Hint: What is A^n for $n \geq 3$?

Solution . . . .

a) Since the singular values are square roots of eigenvalues of $\mathsf{A}^\mathsf{T}\mathsf{A}$, we derive

$$\mathsf{A}^\mathsf{T}\mathsf{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

   and

$$\det(\lambda \mathsf{I} - \mathsf{A}^\mathsf{T}\mathsf{A}) = \lambda(\lambda - 1)(\lambda - 2) - \lambda = \lambda(\lambda^2 - 3\lambda + 1) = 0.$$

   Then $\lambda = 0, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$, and the singular values are $\lambda = 0, \sqrt{\frac{3+\sqrt{5}}{2}}, \sqrt{\frac{3-\sqrt{5}}{2}}$.

b) Since

$$e^{\mathsf{A}t} = \mathsf{I} + \mathsf{A}t + \frac{t^2\mathsf{A}^2}{2!} + \frac{t^3\mathsf{A}^3}{3!} + \cdots,$$

   and

$$\mathsf{A}^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathsf{A}^3 = \mathsf{A}^4 = \cdots = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

   we derive

$$e^{\mathsf{A}t} = \mathsf{I} + \mathsf{A}t + \frac{t^2\mathsf{A}^2}{2!} = \begin{bmatrix} 1 & t & t + \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 4 (30%) ***Positive Definite***

Suppose A is a real m by n Matrix.

a) (10%) Prove that the symmetric matrix $\mathsf{A}^\mathsf{T}\mathsf{A}$ satisfies that

$$\boldsymbol{x}^\mathsf{T}\mathsf{A}^\mathsf{T}\mathsf{A}\boldsymbol{x} \geq 0$$

   for every non-zero vector $\boldsymbol{x}$ in $\mathbb{R}^n$.

b) (10%) According to (a), under what condition on A is $\mathsf{A}^\mathsf{T}\mathsf{A}$ positive definite?

c) (10%) Prove that $\mathsf{A}^\mathsf{T}\mathsf{A}$ is not positive definite if $m < n$.

Solution . . . .

a) $\boldsymbol{x}^\mathsf{T}\mathsf{A}^\mathsf{T}\mathsf{A}\boldsymbol{x} = (\mathsf{A}\boldsymbol{x})^\mathsf{T}(\mathsf{A}\boldsymbol{x}) = (\mathsf{A}\boldsymbol{x}) \cdot (\mathsf{A}\boldsymbol{x}) \geq 0$.

b) From (a), $\mathsf{A}^\mathsf{T}\mathsf{A}$ is not necessarily positive definite because $\boldsymbol{x}^\mathsf{T}\mathsf{A}^\mathsf{T}\mathsf{A}\boldsymbol{x} = (\mathsf{A}\boldsymbol{x}) \cdot (\mathsf{A}\boldsymbol{x})$ might be equal to zero. In order to guarantee $\mathsf{A}^\mathsf{T}\mathsf{A}$ is positive definite, $(\mathsf{A}\boldsymbol{x}) \cdot (\mathsf{A}\boldsymbol{x})$ must not be zero for non-zero $\boldsymbol{x}$. Thus the condition is that A has independent columns.

c) If $m < n$, A cannot have independent columns. Therefore, $\mathsf{A}^\mathsf{T}\mathsf{A}$ is not positive definite according to (b).