2006 Fall Midterm for Advanced Probability for Communications

Each problem costs you 10 points.

1. (Section 6)

   (a) Bernstein's Theorem uses the binomial distribution as a base to construct a polynomial function to uniformly approximate a compact-support continuous function. By following similar technique, find another distribution (other than the binomial one) that yields a polynomial approximate of a compact-support continuous function, which converges uniformly to the desired function. (Hint: Give the approximate polynomial formula and provide the proof.)

   (b) Is there any connection between the Bernstein-like Theorem and the Sampling Theorem in communication area (where a bandlimited function can be distortionlessly re-constructed from its samples). (Hint: The Bernstein-like Theorem is still "valid" even if the distribution of X is not a polynomial function of x.)

Proof:

(a) Denote f to be a continuous function whose support is $[0,1]$. Let $\hat{f}(x) = f(x/2)$. Then, $\hat{f}$ is a continuous function with support $[0,2]$.

Let $X_1, X_2, \ldots$ be discrete i.i.d. random variables taking values from $\{0,1,3\}$ with

$$\begin{cases} \Pr[X_1 = 0] = 1 - x/2, \\ \Pr[X_1 = 1] = x/4, \\ \Pr[X_1 = 3] = x/4 \end{cases}$$

Define

$$S_n = X_1 + \cdots + X_n \quad \text{and} \quad g_n(k,x) = \Pr\left[S_n = k\right].$$

Note that $g_n(k,n)$ is a polynomial of x. Then, $\hat{f}$ can be approximated by

$$B_n(x) = \sum_{k=1}^{2n} g_n(k,x)\,\hat{f}\,(k/n)$$

and this approximate converges to $\hat{f}(x)$ uniformly on $[0,2]$. The proof is as follows.

Let

$$M = \sup_{x\in[0,2]} |\hat{f}(x)|$$

and

$$\delta(\varepsilon) = \sup_{\{(x,y)\in[0,2]\times[0,2]:|x-y|\le\varepsilon\}} |\hat{f}(x)-\hat{f}(y)|.$$

By the continuity of $\hat{f}(\cdot)$ over a closed and bounded (hence, compact) set $[0,2]$,

$$\lim_{\varepsilon\downarrow 0}\delta(\varepsilon) = 0.$$

For convenience, denote $\mathbb{N}_n = \{0,1,2,\ldots,2n\}$. Then

$$\begin{aligned}
\left|B_n(x)-\hat{f}(x)\right| &= \left|E\left[f\left(S_n/n\right)\right]-\hat{f}(x)\right| \\
&= \left|E\left[f\left(S_n/n\right)-\hat{f}(x)\right]\right| \\
&\le E\left[\left|f\left(S_n/n\right)-\hat{f}(x)\right|\right] \\
&= \sum_{k\in\mathbb{N}_n}\left|f\left(k/n\right)-\hat{f}(x)\right|\Pr[S_n=k] \\
&= \sum_{\{k\in\mathbb{N}_n:|k/n-x|<\varepsilon\}}\left|f\left(k/n\right)-\hat{f}(x)\right|\Pr[S_n=k] \\
&\quad + \sum_{\{k\in\mathbb{N}_n:|k/n-x|\ge\varepsilon\}}\left|f\left(k/n\right)-\hat{f}(x)\right|\Pr[S_n=k] \\
&\le \sum_{\{k\in\mathbb{N}_n:|k/n-x|<\varepsilon\}}\delta(\varepsilon)\Pr[S_n=k] \\
&\quad + \sum_{\{k\in\mathbb{N}_n:|k/n-x|\ge\varepsilon\}}(2M)\Pr[S_n=k] \\
&= \delta(\varepsilon)\Pr\left[|S_n/n-x|<\varepsilon\right]+(2M)\Pr\left[|S_n/n-x|\ge\varepsilon\right] \\
&\le \delta(\varepsilon)+(2M)\frac{\mathrm{Var}[X_1]}{n\varepsilon^2} \quad \text{(by Chebyshev's ineq.)} \\
&\le \delta(\varepsilon)+(2M)\frac{E[X_1^2]}{n\varepsilon^2} \\
&\le \delta(\varepsilon)+\frac{18M}{n\varepsilon^2}, \quad (X_1\le 3)
\end{aligned}$$

for which the upper bound is independent of x (and hence, is a uniform bound).

The proof is completed by noting that for $x \in [0,1]$,

$$B_n(2x) = \sum_{k=1}^{2n} g_n(k, 2x)\, f\left(\frac{k}{2n}\right)$$

well-approximate $f(x)$ and converges to $f(x)$ uniformly.

(b) No unique or formal solution. □

2. (Section 9)

(a) If $\Pr[X_n \geq 1 + Y_n \text{ i.o. in } n] = 1$, and $\Pr[Y_n \geq 0 \text{ i.o. in } n] = 1$, then can we say $\Pr[X_n \geq 1 \text{ i.o. in } n] = 1$?

(b) By considering the case in 2(a), is the proof of the last displayed equation of the law of iterated logarithm (cf. slide 9-77) rigorous?

(c) If

$$\Pr[X_n \geq 1 + Y_n \text{ i.o. in } n] = 1,$$

and

$$\Pr[Y_n \geq 0 \text{ for sufficiently large } n] = 1,$$

then can we say $\Pr[X_n \geq 1 \text{ i.o. in } n] = 1$? (Hint: "sufficiently large" means that there exist N such that the argument is true for every $n \geq N$.)

(d) Suppose that $\{X_i\}_{i=1}^{\infty}$ is zero-mean Gaussian i.i.d. with unit variance. Then,

$$\frac{X_1 + \cdots + X_n}{\sqrt{2n \log\log(n)}}$$

is zero-mean Gaussian distributed with variance $2/\log\log(n)$. Hence, it will converge to a degenerated zero-mean Gaussian with zero variance. Does the above fact contradict to the law of iterative logarithm? (Hint: That X and Y are independent Gaussian distributed does not imply that $\max\{X, Y\}$ is Gaussian distributed.)

Proof:

(a) The answer is "not necessary", since the former may occur only when n is odd, while the latter may occur only at n even. For example, let X_n and Y_n be defined over the same probability space $(\Omega = \{0,1\}, \mathcal{F} = 2^\Omega, P)$. Define $Y_{2k}(\omega) = \omega$ and $Y_{2k-1}(\omega) = -2$. Then,

$$\{\omega \in \Omega : Y_n(\omega) \geq 0 \text{ i.o. in } n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \{\omega \in \Omega : Y_k(\omega) \geq 0\} = \{0,1\}$$

since $Y_n(0) \geq 0$ and $Y_n(1) \geq 0$ for every even n. Define $X_n(\omega) = -1$. Then,

$$\{\omega \in \Omega : X_n(\omega) \geq 1 + Y_n(\omega) \text{ i.o. in } n\} = \{0,1\}$$

since $X_n(0) \geq 1 + Y_n(0)$ and $X_n(1) \geq 1 + Y_n(1)$ for every odd n. However,

$$\{\omega \in \Omega : X_n(\omega) \geq 0 \text{ i.o. in } n\} = \emptyset.$$

(b) On slide 9-75, the proof shows that

$$\Pr\left\{ S_{n_k} - S_{n_{k-1}} \geq \left(1 - \frac{1}{\xi}\right) \sqrt{2n_k \log\log(n_k)} \text{ i.o. in } k \right\} = 1.$$

On slide 9-76, the proof shows that

$$\Pr\left\{ S_{n_k} \geq -\frac{2}{\sqrt{\xi}} \sqrt{2n_k \log\log(n_k)} \text{ i.o. in } k \right\} = 1.$$

These two do not necessarily imply the inference on slide 9-77 from 2(a), i.e.,

$$\Pr\left\{ S_{n_{k-1}} \geq \left(1 - \frac{1}{\xi}\right) \sqrt{2n_k \log\log(n_k)} - \frac{2}{\sqrt{\xi}} \sqrt{2n_k \log\log(n_k)} \text{ i.o. in } k \right\} = 1.$$

Hence, this step is **not** rigorous.

(c) Yes. (Comment: From this, you learn how "difficult" to provide a "completely flawless" rigorous proof.)

(d) The stated fact does not contradict to the law of iterative logarithm.

The law of iterative logarithm deals with the **limit supremum** of

$$\frac{X_1 + \cdots + X_n}{\sqrt{2n \log\log(n)}}.$$

In its original form, **limit supremum** is defined as:

$$\lim_{n\to\infty} \sup_{k\geq n} \frac{X_1 + \cdots + X_k}{\sqrt{2k \log\log(k)}}.$$

Thus, even if $\frac{X_1+\cdots+X_k}{\sqrt{2k\log\log(k)}}$ is Gaussian distributed for each k,

$$\sup_{k\geq n} \frac{X_1 + \cdots + X_k}{\sqrt{2k \log\log(k)}}$$

is no longer Gaussian distributed. □

3. (Section 20)

   (a) Fix a probability space $(\Omega = \{0, 1, 2, 3, \cdots\}, \mathcal{F} = \{\emptyset, \Omega\}, P)$. Define a random variable X that is well-defined over this probability space. (Hint: The event $[\omega \in \Omega : X(\omega) \leq x]$ must lie in $\mathcal{F}$ for any real x.)

   (b) For a probability space $(\Omega = \{1, 2, 3, 4, 5, 6\}, \mathcal{F}, P)$, and a random variable satisfying $X(1) = X(3) = X(5) = -1$ and $X(2) = X(4) = X(6) = 1$, find the σ-field generated by X.

   Answer:

   (a) $X(n) = 1$ for all $n \in \Omega$.

   (b) $[\omega \in \Omega : X(\omega) \leq x] = \begin{cases} \emptyset, & x < -1; \\ \{1, 3, 5\}, & -1 \leq x < 1; \\ \Omega, & x > 1 \end{cases}$

   Hence, the smallest σ-field that contains $\emptyset$, $\{1, 3, 5\}$ and Ω is

$$\{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}.$$

4. (Section 21)

(a) Prove that for non-negative random variable X with density $f(x)$,

$$E[X] = \int_0^\infty \Pr[X > t]dt = \int_0^\infty \Pr[X \geq t]dt.$$

(b) Generalize the above formula to random variance X with $\Pr[X \leq x_0] = 0$ for some (possibly negative) x_0. In other words, determine the mean of X in terms of the complementary cdf of X.

(c) Generalize the above formula to determine $E[X \cdot I_{[X \geq x_0]}]$ for arbitrary random variance X (that is possibly statistically unbounded from below), where $I_{[\cdot]}$ is the set indicator function, and $x_0 < 0$.

Proof:

(a)

$$\begin{aligned}
\int_0^\infty \Pr[X > t]dt &= \int_0^\infty \int_t^\infty f(x)dxdt \\
&= \int_0^\infty \int_0^x f(x)dtdx \\
&= \int_0^\infty xf(x)dx \\
&= E[X].
\end{aligned}$$

(b) Let $Y = X - x_0$. Then

$$\begin{aligned}
E[X] &= E[Y] + x_0 \\
&= \int_0^\infty \Pr[Y > t]dt + x_0 \\
&= \int_0^\infty \Pr[X > t + x_0]dt + x_0.
\end{aligned}$$

(c) Again, let $Y = X \cdot I_{[X \geq x_0]}$. Then, for any $x_0 < 0$,

$$\begin{aligned}
E[XI_{[X \geq x_0]}] &= E[Y] \\
&= \int_0^\infty \Pr[Y > t + x_0]dt + x_0 \\
&= \int_0^\infty \Pr[X > t + x_0]dt + x_0.
\end{aligned}$$

5. (Sections 22 & 25)

(a) Theorem 22.1 and the theorem on slide 22-11 state that if $X_1, X_2, \ldots$ are **pair-wise** independent and identically distributed (i.i.d.), whose marginal mean exists, then

$$\frac{S_n}{n} \to E[X_1] \quad \text{with probability 1,}$$

where $S_n = X_1 + X_2 + \cdots + X_n$. However, since the mean of Cauchy distributions does not exist, the above result does not apply. Then, what can we say about the limit

$$\frac{X_1 + \cdots + X_n}{n}$$

for i.i.d. Cauchy variables? (Hint: In what sense S_n/n converges?)

(b) Can we find any constant a such that

$$\frac{X_1 + \cdots + X_n}{n}$$

converges in probability to a for this i.i.d. Cauchy normalized sum?

(c) Show that for i.i.d. variables with symmetric pdf (symmetric with respect to some point x_0), if S_n/n converges in distribution to degenerated distribution at x_0, then S_n/n converges to x_0 in probability.

Answer:

(a) Slide 21-32 shows that the Cauchy distributions are closure under both convolution and scaling operation. Hence,

$$\frac{X_1 + \cdots + X_n}{n}$$

has density

$$\gamma_\alpha(x) = \frac{1}{\pi}\frac{\alpha}{x^2 + \alpha^2} \quad \text{on } \Re,$$

if each X_i has the same density. Hence, S_n/n converges in distribution to Cauchy distribution with parameter α.

(b) The cdf of Cauchy distribution is $\tan^{-1}(x/\alpha)/\pi + 1/2$. Hence,

$$\begin{aligned}
&\Pr[|S_n/n - a| > \varepsilon] \\
&= \left(1 - \frac{1}{\pi}\tan^{-1}\left(\frac{a+\varepsilon}{\alpha}\right) - \frac{1}{2}\right) + \left(\frac{1}{\pi}\tan^{-1}\left(\frac{a-\varepsilon}{\alpha}\right) + \frac{1}{2}\right) \\
&= 1 - \frac{1}{\pi}\tan^{-1}\left(\frac{a+\varepsilon}{\alpha}\right) + \frac{1}{\pi}\tan^{-1}\left(\frac{a-\varepsilon}{\alpha}\right). \qquad (1)
\end{aligned}$$

If S_n/n converges in probability to a, the quantity (1) must reduce to zero as $n \to \infty$, which is not true for any $\varepsilon > 0$. Therefore, there exists no a such that S_n/n converges in probability to a.

(c) Let $F_X(\cdot)$ represent the cdf of variable X. Then,

$$\begin{aligned}
\Pr[|S_n/n - x_0| > \varepsilon] &= 1 - F_{S_n/n}(x_0+\varepsilon) + F_{S_n/n}(x_0-\varepsilon) \\
&= 1 - F_{S_n/n-x_0}(\varepsilon) + F_{S_n/n-x_0}(-\varepsilon) \\
&= 2F_{S_n/n-x_0}(-\varepsilon) \\
&\to 0,
\end{aligned}$$

where the last equality holds due to symmetry, and the last convergence is valid since $-\varepsilon$ is the continuous point of the limiting degenerated cdf.

6. (Section 25)

(a) When will equality hold for

$$\Pr[A \le a] - \Pr[|A - B| \ge b] \le \Pr\left[B < a + b\right]?$$

(b) Given the cdfs of X and Y, and suppose that X and Y have no point mass, what can we say about the cdf of $|X - Y|$ without the knowledge of joint statical relationship between X and Y?

(c) Can we find a bound for $E[|X - Y|]$? (Hint: Problem 4.)

(d) Comment on the goodness of the bound in (c) by examples.

Answer:

(a)

$$
\begin{aligned}
&\Pr[A \le a] - \Pr[|A-B| \ge b] \\
&\quad = \Pr\left[(A \le a) \wedge (|A-B| \ge b)^c\right] - \Pr\left[(|A-B| \ge b) \wedge [A \le a]^c\right] \\
&\quad \le \Pr\left[(A \le a) \wedge (|A-B| \ge b)^c\right] \\
&\quad = \Pr\left[(A \le a) \wedge (|A-B| < b)\right] \\
&\quad = \Pr\left[(A+b \le a+b) \wedge (A-b < B < A+b)\right] \\
&\quad \le \Pr\left[B < a+b\right]
\end{aligned}
$$

So equality holds if, and only if,

$$\Pr\left[(|A-B| \ge b) \wedge [A > a]\right] = 0$$

and

$$\Pr\left[(B < a+b) \wedge ((A > a) \vee (|A-B| \ge b))\right] = 0.$$

Or equivalently,

$$
\begin{aligned}
&\Pr\left[(|A-B| \ge b) \wedge (A > a)\right] = 0 \\
&\Pr\left[(B < a+b) \wedge (A > a)\right] = 0 \\
&\Pr\left[(B < a+b) \wedge (|A-B| \ge b)\right] = 0.
\end{aligned}
$$

Or equivalenty, $[|A-B| \ge b]$, $[A > a]$ and $[B < a+b]$ are statistically disjoint events in the sense that

$$
\begin{aligned}
&\Pr\left([|A-B| \ge b] \vee [A > a] \vee [B < a+b]\right) \\
&\quad = \Pr(|A-B| \ge b) + \Pr(A > a) + \Pr(B < a+b).
\end{aligned}
$$

(b) Let $F_X(\cdot)$, $F_Y(\cdot)$ and $F_Z(\cdot)$ denote the cdfs of X, Y and $Z = |X-Y|$. Then, by using the inequality in (a),

$$1 - F_Z(b) = \Pr[|X-Y| \ge b] \ge \max_{a \in \Re}\{F_X(a) - F_Y(a+b), F_Y(a) - F_X(a+b)\}.$$

(c)

$$
\begin{aligned}
E[Z] &= \int_0^\infty (1 - F_Z(b))db \\
&\ge \int_0^\infty \max_{a \in \Re}\{F_X(a) - F_Y(a+b), F_Y(a) - F_X(a+b)\}db.
\end{aligned}
$$

(d) No formal solution is provided.

For example, if X and Y has the same marginal cdf, then this bound is trivial (and perhaps, useless).

7. (Section 26)

(a) Suppose that a random variable X has bounded support on $[0, 2\pi)$. Then, its cdf can be uniquely determined by $\varphi_X(m)$ for integer m, where $\varphi_X(t)$ is the characteristic function of X. How to reconstruct $\varphi_X(t)$ in terms of $\{\varphi_X(m)\}_{m \text{ integer}}$.

(b) Sampling theorem tells us that a real signal $s(t)$ whose spectrum $S(f)$ is non-negative, symmetric with respect to $f = 0$ and bandlimited to W can be distortionlessly reconstructed by its samples with sampling rate $2W$. Can we determine $\int_a^b |S(f)|df$ for any $-W < a < b < W$ in terms of these samples? (Note: Show that $S(f) = \int_{-\infty}^{\infty} s(t)e^{-i2\pi ft}dt$ is real.)

(c) What does $\int_a^b S(f)df$ represent if $S(f)$ is a power spectrum density?

(d) If θ (in θ, 2θ, 3θ, $\cdots$) in the example of uniformly distributed modulo 1 is a non-zero rational number, what distribution will the counting measure μ_n converge to?

Answer:

(a) By sampling theorem, we can use sinc functions to interpolate $\varphi_X(m)$ to obtain $\varphi_X(t)$. Details are omitted.

(b) First, notice that $\text{Im}\{S(f)\} = [S(f) - S^*(f)]/2 = [S(f) - S(-f)]/2 = 0$, where $S^*(f) = S(-f)$. Hence, $|S(f)| = S(f)$. Then, by treating $S(f)$ as a distribution, we can determine $\int_a^b S(f)df$ using the theorem introduced in class.

(c) Power lies within a Hz and b Hz.

(d) Let $\theta = p/q$ for some integers p and q with $\gcd(p, q) = 1$. Then, $c_m(n) = 1$ when m is a multiple of q, and converges to zero, otherwise. Hence, $c_m = 1$ when m is a multiple of q, and zero, otherwise. As a result, μ is a measure that has mass $1/q$ at point

0, $2\pi/q$, $4\pi/q$, $6\pi/q$, $\cdots$, $2(q-1)\pi/q$. Notably,

$$c_m = \int_{-\infty}^{\infty} e^{imx}\mu(dx) = \frac{1}{q}\sum_{k=0}^{q-1} e^{i2\pi mk/q} = \begin{cases} 1, & m \text{ is a multiple of } q; \\ 0, & \text{otherwise} \end{cases}$$

This example tells us that when we are forced to choose a rational "seed", it is better to choose one that yields "large" q.

8. (Section 27)

   (a) Suppose $\cdots, X_{-3}, X_{-2}, X_{-1}, X_0, X_1, X_2, X_3, \cdots$ are i.i.d. sequence with mean m, variance σ^2 and finite 12th moment. Let $Y_n = \sum_{k=0}^{K} h_k X_{n-k}$. Prove that

   $$\frac{Y_1 + Y_2 + \cdots + Y_n - E[Y_1 + Y_2 + \cdots + Y_n]}{\sqrt{\text{Var}[Y_1 + Y_2 + \cdots + Y_n]}} \Rightarrow N.$$

   (b) What is the limit of $\text{Var}[Y_1 + Y_2 + \cdots + Y_n]/n$?

   (c) Is it fair to say that the output due to stable linear time-invariant filter and i.i.d. input with finite 12th marginal moment satisfies the central limit theorem?

   Answer:

   (a) Firstly, show that $Y_1, Y_2, \cdots$, is α-mixing with $\alpha_n = 0$. Secondly, show that

   $$E[(Y_n - E[Y_n])^{12}] = E\left[\left(\sum_{k=1}^{K} h_k(X_{n-k} - m)\right)^{12}\right] < \infty$$

   Then, apply Theorem 27.4.

   (b) Put $\varphi_Y(k) = E[(Y_1 - E[Y_1])(Y_{1+k} - E[Y_k])]$ and

   $$\Psi_Y(f) = \sum_{k=-\infty}^{\infty} \varphi_Y(k) e^{-i2\pi kf}$$

the autocovariance function and power spectrum density of $Y_1 + Y_2 + \cdots$, respectively. Then, according to Theorem 27.4,

$$\begin{aligned} \frac{1}{n}\text{Var}[Y_1 + \cdots + Y_n] &= \sum_{k=-\infty}^{\infty} \varphi_Y(k) \\ &= \Psi_Y(0) \\ &= \Psi_X(0)|H(0)|^2 \\ &= \sigma^2 \left(\sum_{k=0}^{K} h_k \right)^2, \end{aligned}$$

where $H(f) = \sum_{k=0}^{K} h_k e^{-i2\pi kf}$.

(c) Certainly.

9. (In general)

(a) Prove that the only differentiable function satisfying $f(x+y) = f(x) + f(y)$ is $f(x) = -\alpha x$ for some α.

(b) What is the distribution of a non-negative random variable with differentiable cdf and satisfying $P[X > x+y | X > x] = P[X > y]$?

(c) Let $X_1, X_2, \cdots$ be i.i.d. sequence with marginal distribution in 9(b). Denote $N_t = \max\{n : S_n \le t\}$, where $S_n = X_1 + \cdots + X_n$. Find the distribution of $P[N_t = n]$ for non-negative integer n.

Proof:

(a) Since

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x) - f(x-h)}{h} = \frac{f(h)}{h},$$

we have $f'(x) = \lim_{h \downarrow 0} f(h)/h$ is a constant, independent of x. This completes the proof.

(b) $P[X > x+y] = P[X > x]P[X > y]$ implies $\log P[X > x+y] = \log P[X > x] + \log P[X > y]$. By (a), we know that $\log P[X > x] = -\alpha x$, which implies $P[X > x] = e^{-\alpha x}$.

(c) Event $[N_t = n]$ is equivalent to event $[S_n \le t < S_{n+1}]$. Through this, you shall be able to prove that

$$P[N_t = n] = e^{-\alpha t}\frac{(\alpha t)^n}{n!}.$$

10. (In general) Theorem 23.2 (cf. slide 28-1) states that for i.i.d. $Z_1, Z_2, \cdots$ with $\Pr[Z_1 = 1] = \lambda/n$ and $\Pr[Z_1 = 0] = 1 - \lambda/n$,

$$\Pr[S_n = i] \to e^{-\lambda}\frac{\lambda^i}{i!}, \tag{2}$$

where $S_n = Z_1 + Z_2 + \cdots + Z_n$. Theorem 27.2 states that

$$\frac{S_n - E[S_n]}{\sqrt{\mathrm{Var}[S_n]}} = \frac{S_n - \lambda}{\sqrt{\lambda(1-\lambda/n)}} \Rightarrow N \quad \left(\text{or equivalently, } \frac{S_n - \lambda}{\sqrt{\lambda}} \Rightarrow N\right).$$

Therefore, $\Pr[S_n/\sqrt{\lambda} - \sqrt{\lambda} = i/\sqrt{\lambda} - \sqrt{\lambda}] = \Pr[S_n = i] \to 0$, a contradiction to (2). What's wrong with the above argument?

Proof:

$$\frac{S_n - E[S_n]}{\sqrt{\mathrm{Var}[S_n]}} \Rightarrow N$$

is wrong, because $Z_1, Z_2, \cdots$ does not satisfy the Lindeberg condition!! Let $\tilde{Z}_n = Z_n - \lambda/n$. Then, taking $\epsilon = 1/(2\sqrt{\lambda})$ and $n \geq \max\left\{\frac{4\lambda}{3}, \frac{8\lambda}{\sqrt{17}-1}\right\}$ yields:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{s_n^2}\int_{[|z|\geq\varepsilon s_n]} z^2 dF_{\tilde{Z}_k}(z) \\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{s_n^2}E\left[\tilde{Z}_k^2 I_{[|\tilde{Z}_k|\geq\varepsilon s_n]}\right] \\
&= \lim_{n\to\infty}\frac{1}{\mathrm{Var}[Z_1]}E\left[\tilde{Z}_1^2 I_{[|\tilde{Z}_1|\geq\varepsilon\sqrt{n\mathrm{Var}[Z_1]}]}\right] \\
&= \lim_{n\to\infty}\frac{n}{\lambda(1-\lambda/n)}E\left[(Z_1-\lambda/n)^2 I_{[|Z_1-\lambda/n|\geq\sqrt{(1-\lambda/n)}/2]}\right] \\
&= \lim_{n\to\infty}\frac{n}{\lambda(1-\lambda/n)}\cdot\frac{\lambda}{n}\left(1-\frac{\lambda}{n}\right)^2 \\
&= 1.
\end{aligned}$$