2006 Spring Midterm for Advanced Probability for Communications

Each problem costs you 10 points.

1. (Section 6)

   (a) It is straightforward that if $\{A_i\}_{i=1}^n$ are disjoint with probability one, then

   $$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n P(A_i). \tag{1}$$

   Prove that the converse statement is also true, i.e., (1) implies that $\{A_i\}_{i=1}^n$ are disjoint with probability one.

   (Hint: List all the subset $\mathbb{I}_1, \mathbb{I}_2, \cdots, \mathbb{I}_{2^n-1}$ of $\mathbb{N} \triangleq \{1, 2, 3, \cdots, n\}$ except the empty set. Define $B_j \triangleq \bigcap_{i\in\mathbb{I}_j} A_i - \bigcap_{i\in\mathbb{N}-\mathbb{I}_j} A_i$. Then, find the general representations of $P\left(\bigcup_{i=1}^n A_i\right)$ and $\sum_{i=1}^n P(A_i)$ in terms of $P(B_j)$, respectively.)

   (b) If $\{A_i\}_{i=1}^\infty$ are disjoint, then what can we say about $P(\limsup_{i\to\infty} A_i)$ from the two Borel-Cantelli lemmas?

   (Hint: Problem 1(a) is valid even if n is replaced by ∞.)

   (c) Define $X_i(\omega) = 1$ if $\omega \in A_i$ and zero, otherwise. Does the strong law (i.e., $(1/n)\sum_{i=1}^n X_i$ converges with probability one to some constant) hold for $\{X_i\}_{i=1}^\infty$?

   (Hint: The strong law requires the validity of the first Borel-Cantelli lemma.)

Proof:

(a) That $\{B_j\}_{j=1}^{2^n-1}$ are disjoint and $\bigcup_{i=1}^n A_i = \bigcup_{j=1}^{2^n-1} B_j$ jointly imply that

$$P\left(\bigcup_{i=1}^n A_i\right) = \sum_{j=1}^{2^n-1} P(B_j).$$

Also, $A_i = \bigcup_{\{j\,:\,i\in\mathbb{I}_j\}} B_j$ for $1 \le i \le n$ implies

$$\sum_{i=1}^n P(A_i) = \sum_{i=1}^n \sum_{\{j\,:\,i\in\mathbb{I}_j\}} P(B_j) = \sum_{j=1}^{2^n-1} |\mathbb{I}_j| \cdot P(B_j).$$

The validity of (1) then gives

$$\sum_{j=1}^{2^n-1} (|\mathbb{I}_j| - 1) \cdot P(B_j) = 0.$$

The proof is completed by noting that the sum of nonnegative $(|\mathbb{I}_j| - 1) \cdot P(B_j)$ equal zero implies all terms should equal zero, and hence, $P(B_j)$ is positive only when $|\mathbb{I}_j| = 1$.

(b) Since $\{A_i\}_{i=1}^{\infty}$ are disjoint,

$$\sum_{i=1}^{\infty} P(A_i) = P\left(\bigcup_{i=1}^{\infty} A_i\right) < 1.$$

Then, by the first Borel-Cantelli lemma, $P(\limsup_{i\to\infty} A_i) = 0$.

(c) $P(\limsup_{i\to\infty} A_i) = 0$ implies

$$P\left(\{\omega \ : \ X_i(\omega) = 1 \text{ for finitely many } i\}\right) = 1.$$

Therefore,

$$P\left(\left\{\omega \ : \ \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i(\omega) = 0\right\}\right) = 1,$$

or equivalently,

$$\Pr\left(\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = 0\right) = 1;$$

hence, the strong law holds. □

2. (Section 9)

(a) Fix a random variable Y with probability density function $f_Y(y)$. Prove that for any strictly increasing positive function $g(\cdot)$ with $g(0) = 1$, and for any positive integer k,

$$E[g^k(Y)] \geq \Pr[Y \geq 0] = E[g^k(Y)] \int_0^{\infty} \frac{f_W(y)}{g^k(y)} dy,$$

where

$$f_W(y) \triangleq \frac{g^k(y)f_Y(y)}{\displaystyle\int_{-\infty}^{\infty} g^k(y)f_Y(y)dy} = \frac{g^k(y)f_Y(y)}{E[g^k(Y)]}.$$

(Hint: Upper bound can be proved by Markov inequality. Lower bound can be proved by the definition of $f_W(y)$.)

(b) Prove that

$$0 \le \int_0^{\infty} \frac{f_W(y)}{g^k(y)} dy \le 1.$$

Proof:

(a) By Markov inequality,

$$\Pr[Y \ge 0] = \Pr[g(Y) \ge g(0)] = \Pr[g(Y) \ge 1] \le \frac{E[g^k(Y)]}{1^k} = E[g^k(Y)].$$

On the other hand,

$$\Pr[Y \ge 0] = \int_0^{\infty} f_Y(y)dy = \int_0^{\infty} E[g^k(Y)]\frac{f_W(y)}{g^k(y)}dy = E[g^k(Y)] \int_0^{\infty} \frac{f_W(y)}{g^k(y)}dy.$$

(b) Since $g(y)$ is a positive function, $E[g^k(Y)] > 0$ (This must be mentioned in the proof!). Hence, by (a), we obtain that

$$\int_0^{\infty} \frac{f_W(y)}{g^k(y)}dy \le 1.$$

The nonnegativity of the integral can be verified by noting that $f_W(y)/g^k(y) \ge 0$ for all y. □

3. (Section 20)

(a) Fix a probability space $(\Omega = \{0, 1, 2, 3, \cdots\}, \mathcal{F} = \{\emptyset, \Omega\}, P)$. Define a random variable X that is well-defined over this probability space. (Hint: The event $[\omega \in \Omega : X(\omega) \le x]$ must lie in $\mathcal{F}$ for any real x.)

(b) For a probability space $(\Omega = \{1, 2, 3, 4, 5, 6\}, \mathcal{F}, P)$, and a random variable satisfying $X(1) = X(3) = X(5) = -1$ and $X(2) = X(4) = X(6) = 1$, find the σ-field generated by X.

Answer:

(a) $X(n) = 1$ for all $n \in \Omega$.

(b) $[\omega \in \Omega : X(\omega) \le x] = \begin{cases} \emptyset, & x < -1; \\ \{1,3,5\}, & -1 \le x < 1; \\ \Omega, & x > 1 \end{cases}$ Hence, the smallest σ-field that contains $\emptyset$, $\{1,3,5\}$ and Ω is $\{\emptyset, \{1,3,5\}, \{2,4,6\}, \Omega\}$.

4. (Section 21)

(a) Prove the generalized Hölder's inequality, i.e., for $p_j > 0$ with $\sum_{j=1}^n 1/p_j = 1$,

$$E\left[\left|\prod_{j=1}^n X_j\right|\right] \le \prod_{j=1}^n E^{1/p_j}\left[|X_j|^{p_j}\right].$$

(Note: Remember to give the condition under which equality holds.)

(b) How to modify the generalized Hölder's inequality if $\sum_{j=1}^n 1/p_j \neq 1$ (but still $p_j > 0$ for all j).

Proof:

(a) Since the inequality is trivially valid, if $\prod_{j=1}^n E^{1/p_j}\left[|X|^{p_j}\right] = 0$. Without loss of generality, assume $\prod_{j=1}^n E^{1/p_j}\left[|X|^{p_j}\right] > 0$.

- $\exp\{x\}$ is a convex function in x. Hence, by Jensen's inequality,

$$\exp\left\{\sum_{j=1}^n \frac{1}{p_j} s_j\right\} \le \sum_{j=1}^n \frac{1}{p_j} \exp\{s_j\}.$$

Since e^x is strictly convex, equality holds iff $s_j = s$ for all j.

- Let $a_j = \exp\{s_j/p_j\}$. Then the above inequality becomes:

$$\prod_{j=1}^n a_j \le \sum_{j=1}^n \frac{1}{p_j} a_j^{p_j},$$

Equality holds iff $a_j^{p_j} = e^s$ for all j.

whose validity is not restricted to positive a_j but to non-negative a_j.

- By letting $a_j = |X_j|/E^{1/p_j}[|X_j|^{p_j}]$, we obtain:

$$\frac{|\prod_{j=1}^n X_j|}{\prod_{j=1}^n E^{1/p_j}[|X_j|^{p_j}]} \le \sum_{j=1}^n \frac{1}{p_j} \frac{|X_j|^{p_j}}{E[|X_j|^{p_j}]}.$$

  Taking the expectation values of both sides yields:

$$\frac{E[|\prod_{j=1}^n X_j|]}{\prod_{j=1}^n E^{1/p_j}[|X_j|^{p_j}]} \le \sum_{j=1}^n \frac{1}{p_j} = 1.$$

  Equality holds iff $\Pr\left[\frac{|X_j|^{p_j}}{E[|X_j|^{p_j}]} = \text{constant for all } j\right] = 1$.

(b) Suppose $\sum_{j=1}^n 1/p_j = q > 0$. Then $\sum_{j=1}^n 1/(p_j q) = 1$.

$$E\left[\left|\prod_{j=1}^n X_j\right|\right] \le \prod_{j=1}^n E^{1/(p_j q)}\left[|X_j|^{(p_j q)}\right].$$

5. (Section 22) Prove that equality holds for the inequality in Theorem 22.4 (slide 22-19) only when both sides are either one or zero.

   Proof: From the proof, we found that equality holds if

$$E\left[S_n^2(I_{A_{n+1}} + I_{A_{n+2}} + \cdots)\right] = 0 \tag{2}$$
$$\sum_{k=1}^n E\left[(S_n - S_k)^2 I_{A_k}\right] = E\left[\sum_{k=1}^n (S_n - S_k)^2 I_{A_k}\right] = 0 \tag{3}$$
$$\sum_{k=1}^n E[(S_k^2 - \alpha^2) I_{A_k}] = 0. \tag{4}$$

   Since $S_n^2(I_{A_{n+1}} + I_{A_{n+2}} + \cdots)$ is non-negative, (2) tells that $S_n^2(I_{A_{n+1}} + I_{A_{n+2}} + \cdots) = 0$ with probability 1, which implies either $\max_{1\le k\le n}|S_k| \ge \alpha$ or $S_n = 0$. We discuss the two conditions separately below.

   - The condition $\max_{1\le k\le n}|S_k| \ge \alpha$ implies that exact one of $\{I_{A_k}\}_{k=1}^n$ is one. Let $I_{A_j} = 1$ for some $1 \le j \le n$. Then, (3) and (4) indicate that $(S_n - S_j)^2 = 0$ and $S_j^2 = \alpha^2$, i.e, $S_n = S_j$ and $\Pr(S_j = \alpha) = \Pr(S_j = -\alpha) = 1/2$ for some $1 \le j \le n$ satisfying

$I_{A_j} = 1$. Thus, the first condition that makes the inequality in Theorem 22.4 equal is:

$$\exists j \text{ among } 1 \le j \le n \text{ such that } \max_{1 \le k \le j-1} |S_k| < \alpha \text{ and } S_n = S_j$$

$$\text{and } \Pr(S_j = \alpha) = \Pr(S_j = -\alpha) = \frac{1}{2},$$

a case that makes both sides of the inequality one!

- The condition $S_n = 0$ reduces (3) and (4) to $\sum_{k=1}^n E[S_k^2 I_{A_k}] = 0$ and $\sum_{k=1}^n \alpha^2 E[I_{A_k}] = 0$. Hence, $I_{A_k} = 0$ with probability 1 for every $1 \le k \le n$. Thus, the second situation that equates the inequality in Theorem 22.4 is

$$S_n = 0 \text{ and } \max_{1 \le k \le n-1} |S_k| < \alpha,$$

a trivial case that makes both sides of the inequality zero!

6. (Section 25) Based on Theorem 25.11, give an example that $X_n \Rightarrow X$ and $E[|X|] < \liminf_{n\to\infty} E[|X_n|]$. (Hint: Let $X_n = X + \delta_n Y$ for non-negative X, Y and δ_n. Then define the required properties of X, Y and constant sequence δ_n to satisfy the two conditions. Remember to use "Properties regarding convergence in distribution" to prove that $X_n \Rightarrow X$.)

Answer: Find non-negative X and Y such that $E[X] < \infty$ and $E[Y] = \infty$. Also, find a positive sequence $\{\delta_n\}_{n=1}^\infty$ satisfying $\delta_n \to 0$.

1. $X_n \Rightarrow X$**:** From slide 25-15, $Y \Rightarrow Y$ and $\delta_n \to 0$ implies $\delta_n Y \Rightarrow 0$. From Theorem 25.4, $X \Rightarrow X$ and $X - (X + \delta_n Y) \Rightarrow 0$ implies $X + \delta_n Y \Rightarrow X$.

2. $E[|X|] < \liminf_{n\to\infty} E[|X_n|]$**:**

$$\begin{aligned} E[|X|] &= E[X] \\ &< \liminf_{n\to\infty} E[|X_n|] \\ &= \liminf_{n\to\infty} E[X_n] \\ &= \liminf_{n\to\infty} (E[X] + \delta_n E[Y]) \\ &= \infty. \end{aligned}$$

7. (Section 26)

(a) Point out how to refine the proof on slide 26-52∼54 so that the theorem statement can be improved to:

Theorem Suppose the support of the distribution of random variable X is contained in $[0, 2\pi]$. Then

$$\frac{1}{2}\Pr[X = a] + \Pr[a < X < b] + \frac{1}{2}\Pr[X = b] = \lim_{m\to\infty}\int_a^b \sigma_m(t)dt,$$

if $0 < a < b < 2\pi$, where

$$\sigma_m(t) = \frac{1}{2\pi m}\int_0^{2\pi} \frac{\sin^2[m(x-t)/2]}{\sin^2[(x-t)/2]}dF_X(x).$$

(Hint: $\sin^2(ms/2)/\sin^2(s/2)$ is an even function in s.)

(b) Based on the above theorem, we have the next corollary.

Corollary Suppose the support of the distribution of random variable X is contained in $[-\pi, \pi]$. Let $f_X(x)$ be the density of X. Then

$$\int_a^b f_X(x)dx = \lim_{m\to\infty}\int_a^b \sigma_m(t)dt,$$

if $-\pi < a < b < \pi$, where $\sigma_m(t) = \frac{1}{2\pi m}\sum_{\ell=0}^{m-1}\sum_{k=-\ell}^{\ell} c_k e^{-itk}$, $c_m = \varphi_X(m)$, and $\varphi_X(t) = \int_0^{2\pi} e^{itx}dF_X(x)$.

Now let a (possibly complex) function $h(t)$ have non-negative bounded real spectrum $H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}dt$ satisfying $H(f) = 0$ for $|f| > \pi$. By the above corollary, represent $\int_a^b H(f)df$ in terms of the samples of $h(t)$, where $-\pi < a < b < \pi$. (Note: No integration operation is allowed to leave unresolved in the answer.)

(c) In (b), if we further assume that $H(f)$ is symmetric (i.e., $H(f) = H(-f)$; hence, $h(t)$ is real), then show that $\int_a^b H(f)df$ can be re-formulated in the form of **sinc** function.

Proof:

(a) Since $\sin^2(ms/2)/\sin^2(s/2)$ is a bounded even function,

$$\frac{1}{2\pi m}\int_{-\pi}^{0}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds = \frac{1}{2\pi m}\int_{0}^{\pi}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds.$$

Hence,

$$\frac{1}{2\pi m}\int_{-\pi}^{0}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds = \frac{1}{2\pi m}\int_{0}^{\pi}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds = \frac{1}{2},$$

which implies that

$$\begin{aligned}\lim_{m\to\infty}\frac{1}{2\pi m}\left(\int_{a-x}^{b-x}\frac{\sin^2(ms/2)}{\sin^2(s/2)}ds\right) &= \begin{cases} 0, & \text{if } a-x>0; \\ \frac{1}{2}, & \text{if } a-x=0; \\ 1, & \text{if } a-x<0<b-x; \\ \frac{1}{2}, & \text{if } b-x=0; \\ 0, & \text{if } b-x<0 \end{cases} \\ &= \begin{cases} 0, & \text{if } x<a \text{ or } x>b; \\ \frac{1}{2}, & \text{if } x=a \text{ or } x=b; \\ 1, & \text{if } a<x<b. \end{cases}\end{aligned}$$

Accordingly, the new theorem statement holds.

(b) Let the density of X in (a) be $f_X(x) = H(x)/\int_{-\pi}^{\pi}H(f)df$. Then, $h(t) = \int_{-\infty}^{\infty}H(f)e^{i2\pi ft}df = \left(\int_{-\pi}^{\pi}H(f)df\right)\left(\int_{-\infty}^{\infty}f_X(x)e^{i2\pi tx}dx\right) = \left(\int_{-\pi}^{\pi}H(f)df\right)\varphi_X(2\pi t)$, which implies

$$c_m = \varphi_X(m) = h(m/(2\pi))/\left(\int_{-\pi}^{\pi}H(f)df\right).$$

Hence,

$$\begin{aligned}\sigma_m(t) &= \frac{1}{2\pi m}\sum_{\ell=0}^{m-1}\sum_{k=-\ell}^{\ell}c_k e^{-itk} \\ &= \frac{1}{2\pi m\left(\int_{-\pi}^{\pi}H(f)df\right)}\sum_{\ell=0}^{m-1}\sum_{k=-\ell}^{\ell}h\left(\frac{k}{2\pi}\right)e^{-itk}\end{aligned}$$

and

$$\begin{aligned}
\frac{\int_a^b H(f)df}{\int_{-\pi}^{\pi} H(f)df} &= \lim_{m\to\infty} \int_a^b \sigma_m(t)dt \\
&= \frac{1}{2\pi\left(\int_{-\pi}^{\pi} H(f)df\right)} \lim_{m\to\infty} \frac{1}{m} \int_a^b \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} h\left(\frac{k}{2\pi}\right) e^{-itk} dt \\
&= \frac{1}{2\pi\left(\int_{-\pi}^{\pi} H(f)df\right)} \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} h\left(\frac{k}{2\pi}\right) \int_a^b e^{-itk} dt \\
&= \frac{1}{2\pi\left(\int_{-\pi}^{\pi} H(f)df\right)} \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} h\left(\frac{k}{2\pi}\right) \frac{e^{-ika} - e^{-ikb}}{ik}.
\end{aligned}$$

Consequently,

$$\int_a^b H(f)df = \lim_{m\to\infty} \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} h\left(\frac{k}{2\pi}\right) \frac{e^{-ika} - e^{-ikb}}{ik}.$$

(c) Observe that

$$\begin{aligned}
\int_{-b}^b H(f)df &= \lim_{m\to\infty} \frac{1}{2\pi m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} h\left(\frac{k}{2\pi}\right) \frac{e^{ikb} - e^{-ikb}}{ik} \\
&= \lim_{m\to\infty} \frac{1}{m} \sum_{\ell=0}^{m-1} \sum_{k=-\ell}^{\ell} h\left(\frac{k}{2\pi}\right) \frac{\sin(kb)}{\pi k}.
\end{aligned}$$

Then the problem can be solved by noting that

$$2\int_a^b H(f)df = \begin{cases} \int_{-b}^b H(f)df - \int_{-a}^a H(f)df, & 0 < a < b; \\ \int_{-b}^b H(f)df + \int_a^{-a} H(f)df, & a < 0 < b; \\ \int_b^{-b} H(f)df - \int_a^{-a} H(f)df, & a < b < 0; \end{cases}$$

8. (Section 27) Show that if $z_1, \ldots, z_m$ and $w_1, \ldots, w_m$ are complex numbers, not necessarily of modulus at most 1, then

$$|z_1 \times z_2 \times \cdots \times z_m - w_1 \times w_2 \times \cdots \times w_m| \leq \sum_{k=1}^m |z_k - w_k|$$

may not be true. (Hint: Give an example of z_i and w_i that are not of modulus at most 1, and show that the inequality does not hold.)

Answer: Take $m = 2$, we can find that in the proof, only the last step requires the condition of "modulus at most 1". Specifically,

$$\begin{aligned} & |z_1 \times z_2 - w_1 \times w_2| \\ = & |(z_1 - w_1)z_2 + w_1(z_2 - w_2)| \\ \leq & |(z_1 - w_1)z_2| + |w_1(z_2 - w_2)| \\ = & |z_1 - w_1||z_2| + |w_1|\,|z_2 - w_2| \\ \leq & |z_1 - w_1| + |z_2 - w_2|\,. \end{aligned}$$

A counterexample is easy to construct by taking $w_i = 0$ for all i, and $|z_1 \times z_2 \times \cdots z_m| > \sum_{k=1}^{m} |z_k|$, e.g., $z_k = k$ for all k.

9. (In general)

   (a) Prove that $E[X] = \int_0^\infty \Pr[X > t]dt$ for non-negative bounded random variable X with density $f(x)$. (Hint: Fubini's Theorem.)

   (b) Using (a) to prove that $E[X] = \int_0^\infty \Pr[X > t]dt$ for non-negative (possibly unbounded) random variable X with density $f(x)$. (Hint: Define $Y_n = XI_{[x \leq n]}$, and Fatou's lemma tells that

   $$E[X] \leq \liminf_{n\to\infty} E[Y_n] \quad \text{and} \quad \int_0^\infty \Pr[X > t]dt \leq \liminf_{n\to\infty} \int_0^\infty \Pr[Y_n > t]dt.)$$

Proof:

(a) Denote by M the bound for random variable X, i.e., $\Pr[X \leq M] = 1$. By Fubini's Theorem (the condition for Fubini's Theorem is always valid for bounded integration region),

$$\begin{aligned} \int_0^\infty \Pr[X > t]dt &= \int_0^M \Pr[X > t]dt \\ &= \int_0^M \left(\int_t^M f(x)dx \right) dt \\ &= \int_0^M \left(\int_0^x f(x)dt \right) dx \\ &= \int_0^M xf(x)dx \\ &= E[X]. \end{aligned}$$

(b) Define $Y_n = XI_{[X \le n]}$. Then $\Pr[Y_n > t] \to \Pr[X > t]$ as $n \to \infty$, or equivalently, $Y_n \Rightarrow X$.

By Fatou's Lemma,

$$E[Y_n] \le E[X] \le \liminf_{n \to} E[Y_n].$$

Therefore, $E[X] = \lim_{n \to \infty} E[Y_n]$.

On the other hand,

$$\int_0^\infty \Pr[Y_n > t]dt \le \int_0^\infty \Pr[X > t]dt \le \liminf_{n \to \infty} \int_0^\infty \Pr[Y_n > t]dt,$$

which implies that

$$\int_0^\infty \Pr[X > t]dt = \lim_{n \to \infty} \int_0^\infty \Pr[Y_n > t]dt.$$

The proof is completed by noting that $E[Y_n] = \int_0^\infty \Pr[Y_n > t]dt$ from (a). □

10. (In general) A random variable has a *lattice distribution* if for some a and b, where $b \ne 0$, the set $\{a + nb \ : \ n = 0, \pm 1, \pm 2, \ldots\}$ supports the distribution of X.

    (a) Prove that X has a lattice distribution, if its characteristic function $|\varphi_X(t)| = 1$ for some $t \ne 0$.
    (Hint: A complex value $\varphi_X(t_0)$ satisfying $|\varphi_X(t_0)| = 1$ can be represented as $\varphi_X(t_0) = e^{it_0 a}$ for some a. That a non-negative function like $[1 - \cos(it_0(x - a))]$ has integral zero implies that the function is exactly zero in the integration domain.)

    (b) Prove that if X has a lattice distribution, then its characteristic function $|\varphi_X(t)| = 1$ for some $t \ne 0$.
    (Hint: Let $b = 2\pi/t_0$.)

Proof:

(a) If $|\varphi_X(t_0)| = 1$ for some $t_0 \neq 0$, then $\varphi_X(t_0) = e^{it_0a}$ for some a.

$$\begin{aligned}
0 &= 1 - e^{-it_0a}\varphi_X(t_0) \\
&= \int_{-\infty}^{\infty} dF_X(x) - e^{-it_0a}\int_{-\infty}^{\infty} e^{it_0x}dF_X(x) \\
&= \int_{-\infty}^{\infty} (1 - e^{it_0(x-a)})dF_X(x) \\
&= \int_{-\infty}^{\infty} [1 - \cos(it_0(x-a))]dF_X(x) + i\int_{-\infty}^{\infty} \sin(it_0(x-a))dF_X(x).
\end{aligned}$$

As $[1-\cos(it_0(x-a))] \geq 0$ and $\int_{-\infty}^{\infty}[1-\cos(it_0(x-a))]dF_X(x) = 0$, we conclude that

$$\Pr[\cos(it_0(X-a)) = 1] = 1,$$

which implies that

$$\Pr[t_0(X-a) = 2n\pi \text{ for } n = 0, \pm 1, \pm 2, \ldots] = 1.$$

(b) Let $b = 2\pi/t_0$.

$$\begin{aligned}
\varphi_X(t_0) &= \sum_{n=0,\pm1,\pm2,\ldots} e^{it_0(a+nb)}p_n \\
&= \sum_{n=0,\pm1,\pm2,\ldots} e^{it_0(a+2\pi n/t_0)}p_n \\
&= e^{it_0a}\sum_{n=0,\pm1,\pm2,\ldots} e^{i2\pi n}p_n \\
&= e^{it_0a}.
\end{aligned}$$

So $|\varphi_X(t_0)| = 1$. □