**2016 Final for Advanced Probability for Communications**

The number of total points is 100 in this exam.

1. (8 pt.) The Law of the Iterated Logarithm states that for i.i.d. $\{X_i\}_{i=1}^{\infty}$ with mean 0 and variance 1,

$$\Pr\left[\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log(n)}} = 1\right] = 1$$

where $S_n = X_1 + \cdots + X_n$. Then for arbitrarily small $\epsilon > 0$, what is the limiting probability for

$$\limsup_{n\to\infty} \Pr\left[\left|\frac{S_n}{\sqrt{2n\log\log(n)}} - 1\right| > \epsilon\right]?$$

Justify your ansewr.

Hint: $S_n/\sqrt{n} \Rightarrow N$, where $N$ is standard normal distributed.

**Solution.** The central limit theorems implies that

$$S_n/\sqrt{n} \Rightarrow N.$$

Thus,

$$\begin{aligned}
&\limsup_{n\to\infty} \Pr\left[\left|\frac{S_n}{\sqrt{2n\log\log(n)}} - 1\right| > \epsilon\right] \\
&= \limsup_{n\to\infty} \left(\Pr\left[\frac{S_n}{\sqrt{n}} > (1+\epsilon)\sqrt{2\log\log(n)}\right] + \Pr\left[\frac{S_n}{\sqrt{n}} < (1-\epsilon)\sqrt{2\log\log(n)}\right]\right) \\
&\geq \limsup_{n\to\infty} \Pr\left[\frac{S_n}{\sqrt{n}} < (1-\epsilon)\sqrt{2\log\log(n)}\right] \\
&\geq \limsup_{n\to\infty} \Pr\left[\frac{S_n}{\sqrt{n}} < L\right] \\
&= \Phi(L).
\end{aligned}$$

Since $L$ can be arbitrarily large,

$$\limsup_{n\to\infty} \Pr\left[\left|\frac{S_n}{\sqrt{2n\log\log(n)}} - 1\right| > \epsilon\right] = 1.$$

2. Define a cdf $F$ recursively via the following four rules.

   1) $F(0) = 0$ and $F(1) = 1$;
   2) $F(x) = \frac{1}{2}$ for $\frac{1}{3} < x < \frac{2}{3}$;
   3) $F(x/3) = \frac{1}{2}F(x)$ for $0 \le x \le 1$;
   4) $F(1-x) = 1 - F(x)$.

   By the rules, we obtain

$$F\left(\tfrac{1}{3}\right) = \tfrac{1}{2}F(1) = \tfrac{1}{2}; \quad F\left(\tfrac{2}{3}\right) = 1 - F\left(\tfrac{1}{3}\right) = \tfrac{1}{2},$$

   and

$$\begin{array}{ll} F\left(\frac{1}{9}\right) = \frac{1}{2}F\left(\frac{1}{3}\right) = \frac{1}{4}; & F\left(\frac{2}{9}\right) = \frac{1}{2}F\left(\frac{2}{3}\right) = \frac{1}{4}; \\ F\left(\frac{3}{9}\right) = F\left(\frac{1}{3}\right) = \frac{1}{2}; & F\left(\frac{4}{9}\right) = \frac{1}{2}; \\ F\left(\frac{5}{9}\right) = 1 - F\left(\frac{4}{9}\right) = \frac{1}{2}; & F\left(\frac{6}{9}\right) = 1 - F\left(\frac{3}{9}\right) = \frac{1}{2}; \\ F\left(\frac{7}{9}\right) = 1 - F\left(\frac{2}{9}\right) = \frac{3}{4}; & F\left(\frac{8}{9}\right) = 1 - F\left(\frac{1}{9}\right) = \frac{3}{4}. \end{array}$$

   We can continue the above procedure to obtain the $F$-function values for $x = k/3^\ell$ for every $\ell \ge 3$ and $0 \le k \le 3^\ell$.

   Based on the setting, the Riemann upper and lower approximations of $\int_0^1 x\, dF(x)$ with $\Delta x = 3^{-2}$ is given by:

$$\begin{aligned} L_2 &\triangleq \sum_{i=0}^{8} \left(\frac{i}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] \\ &\le \int_0^1 x\, dF(x) \le \sum_{i=0}^{8} \left(\frac{i+1}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] \triangleq U_2, \end{aligned}$$

   where

$$L_2 = \sum_{i=0}^{8} \left(\frac{i}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] = \frac{(0+2+6+8)}{9} \cdot \frac{1}{4} = \frac{4}{9}$$

   and

$$U_2 = \sum_{i=0}^{8} \left(\frac{i+1}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] = \frac{(1+3+7+9)}{9} \cdot \frac{1}{4} = \frac{5}{9}.$$

The Riemann upper and lower approximations of $\int_0^1 x\,dF(x)$ with $\Delta x = 3^{-3}$ is given by

$$\begin{aligned} L_3 &\triangleq \sum_{i=0}^{26} \left(\frac{i}{27}\right) \left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \\ &\le \int_0^1 x\,dF(x) \le \sum_{i=0}^{26} \left(\frac{i+1}{27}\right) \left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \triangleq U_3, \end{aligned}$$

where

$$\begin{aligned} L_3 &= \sum_{i=0}^{26} \left(\frac{i}{27}\right) \left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \\ &= \frac{(0+2+6+8)+(18+20+24+26)}{27} \cdot \frac{1}{8} = \frac{13}{27} \end{aligned}$$

and

$$\begin{aligned} U_3 &= \sum_{i=0}^{26} \left(\frac{i+1}{27}\right) \left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \\ &= \frac{(1+3+7+9)+(19+21+25+27)}{27} \cdot \frac{1}{8} = \frac{14}{27}. \end{aligned}$$

Noting that $F\left(\frac{i+1}{3^\ell}\right) - F\left(\frac{i}{3^\ell}\right)$ is either 0 or $2^{-\ell}$, answer the following questions.

(a) (8 pt.) Find the Riemann approximation of $\int_0^1 x\,dF(x)$ with $\Delta x = 3^{-\ell}$. In other words, determine the upper and lower bounds of the below equation:

$$\begin{aligned} L_\ell &\triangleq \sum_{i=0}^{3^\ell-1} \left(\frac{i}{3^\ell}\right) \left[F\left(\frac{i+1}{3^\ell}\right) - F\left(\frac{i}{3^\ell}\right)\right] \\ &\le \int_0^1 x\,dF(x) \le \sum_{i=0}^{8} \left(\frac{i+1}{3^\ell}\right) \left[F\left(\frac{i+1}{3^\ell}\right) - F\left(\frac{i}{3^\ell}\right)\right] \triangleq U_\ell. \end{aligned}$$

Hint: Based on $L_2$ and $L_3$, deduce the formula of $L_\ell$. Then deduce the relation between $L_\ell$ and $U_\ell$.

(b) (8 pt.) Determine $\lim_{\ell\to\infty} L_\ell$ and $\lim_{\ell\to\infty} U_\ell$ and check whether they are equal or not.

**Solutions.**

(a)

$$L_2 = \frac{(3^2-1)\cdot 2}{3^2\cdot 2^2} \text{ and } U_2 = L_2 + \frac{2^2}{3^2\cdot 2^2}$$
$$L_3 = \frac{(3^3-1)\cdot 2^2}{3^3\cdot 2^3} \text{ and } U_3 = L_3 + \frac{2^3}{3^3\cdot 2^3}$$
$$\cdots$$
$$L_\ell = \frac{(3^\ell-1)\cdot 2^{\ell-1}}{3^\ell\cdot 2^\ell} = \frac{1}{2}(1-3^{-\ell}) \text{ and } U_\ell = L_\ell + \frac{2^\ell}{3^\ell\cdot 2^\ell} = \frac{1}{2}(1+3^{-\ell})$$

(b) Apparently, $\lim_{\ell\to\infty} L_\ell = \lim_{\ell\to\infty} U_\ell = \frac{1}{2}$.

3. (a) (8 pt.) Let $\{P_i\}_{i=1}^m$ and $\{Q_i\}_{i=1}^m$ be positive real numbers satisfying

$$\sum_{i=1}^m P_i = \sum_{i=1}^m Q_i = 1.$$

Prove that for $0 < \lambda < 1$,

$$\sum_{i=1}^m Q_i^\lambda P_i^{1-\lambda} \le 1,$$

and give a necessary and sufficient condition for equality.

Hint: Note that $\sum_{i=1}^m Q_i^\lambda P_i^{1-\lambda} = \sum_{i=1}^m P_i \left(\frac{Q_i}{P_i}\right)^\lambda = E[f(X)]$, where $f(x) = x^\lambda$ and $X$ is a random variable with alphabet

$$\left\{\frac{Q_1}{P_1}, \frac{Q_2}{P_2}, \ldots, \frac{Q_m}{P_m}\right\}$$

and distribution $\Pr[X = Q_i/P_i] = P_i$. Then apply Jensen's inequality on $E[f(X)]$.

(b) (8 pt.) Continue from (a). If $\lambda > 1$ or $\lambda < 0$, does

$$\sum_{i=1}^{m} Q_i^{\lambda} P_i^{1-\lambda} \le 1$$

still hold ? Justify your answer.

(c) (10 pt.) (General Hölder's inequality) Prove that for positive real numbers $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$,

$$\sum_{i=1}^{m} a_i b_i \begin{cases} \le \left(\sum_{i=1}^m a_i^{1/\lambda}\right)^{\lambda} \left(\sum_{i=1}^m b_i^{1/(1-\lambda)}\right)^{1-\lambda}, & 0 < \lambda < 1; \\ \ge \left(\sum_{i=1}^m a_i^{1/\lambda}\right)^{\lambda} \left(\sum_{i=1}^m b_i^{1/(1-\lambda)}\right)^{1-\lambda}, & \lambda < 0 \text{ or } \lambda > 1; \end{cases}$$

with equality holding if, and only if, for some constant $c$, $a_i^{1/\lambda} = c b_i^{1/(1-\lambda)}$ for all $i$.

Hint: Use (a) and (b) with properly setting $P_i$ and $Q_i$. For example, you may set $P_i = \frac{b_i^{1/\lambda}}{\sum_{i=1}^m b_i^{1/\lambda}}$.

**Solutions.**

(a) Noting that $f''(x) = \lambda(\lambda - 1)x^{\lambda-2} < 0$ for $x > 0$ and $0 < \lambda < 1$, we obtain from Jensen's inequality that

$$E[f(X)] \le f(E[X]) = \left(\sum_{i=1}^m P_i \frac{Q_i}{P_i}\right)^{\lambda} = \left(\sum_{i=1}^m Q_i\right)^{\lambda} = 1.$$

Since $f(x)$ is strictly concave, equality holds if, and only if, $X$ is deterministic, i.e., $P_i = Q_i$ for every $i$.

(b) $f''(x) = \lambda(\lambda - 1)x^{\lambda-2} > 0$ for $x > 0$ and ($\lambda < 0$ or $\lambda > 1$). Thus

$$E[f(X)] \ge f(E[X]) = \left(\sum_{i=1}^m P_i \frac{Q_i}{P_i}\right)^{\lambda} = \left(\sum_{i=1}^m Q_i\right)^{\lambda} = 1.$$

(c) Setting

$$Q_i = \frac{a_i^{1/\lambda}}{\sum_{i=1}^m a_i^{1/\lambda}} \quad \text{and} \quad P_i = \frac{b_i^{1/\lambda}}{\sum_{i=1}^m b_i^{1/\lambda}}$$

yields the desired result.

4. (a) (8 pt.) Suppose that $\{X_n\}_{n=1}^{\infty}$ is an independent sequence of random variables having the same distribution $\Pr[X_i = -1] = \Pr[X_i = 1] = \frac{1}{2}$. Then Theorem 27.1 states that

$$\frac{S_n}{\sqrt{n}} \Rightarrow N,$$

where $N$ is standard normal distributed, and $S_n = X_1 + \cdots + X_n$. Find the fourth moment of $S_n/\sqrt{n}$ using cumulant formulas. Is it asymptotically equal to $E[N^4]$ as $n$ goes to infinity?

Hint: The cumulants of a zero-mean random variable $X$ are given by

$$\begin{aligned} C^{(1)}(0) = c_1 &= 0 \\ C^{(2)}(0) = c_2 &= E[X^2] = \mathrm{Var}[X] \\ C^{(3)}(0) = c_3 &= E[X^3] \\ C^{(4)}(0) = c_4 &= E[X^4] - 3E^2[X^2] \\ &\vdots \end{aligned}$$

Recall why these terms are named "cumulants." Also, $E[N^2] = 1$ and $E[N^4] = 3$.

(b) (8 pt.) Is it always true that for i.i.d. zero-mean and unit variance random variables $\{X_i\}_{i=1}^{\infty}$,

$$\lim_{n\to\infty} E[S_n^4/n^2] = E[N^4] = 3.$$

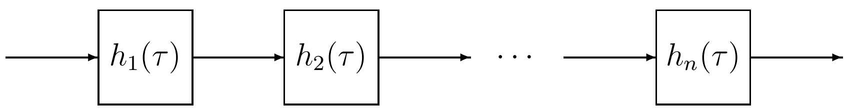
If your answer is affirmative, prove it. If not, justify your answer.

Hint: Do we assume that $X_i$ has finite fourth moment?

(c) (8 pt.) For the tandem filtering bank with

$$h_1(\tau) = h_2(\tau) = \cdots = h_n(\tau) = \sqrt{n} \cdot g(\sqrt{n}\tau)$$

satisfying $G(0) = 1$, $G'(0) = 0$ and $G''(0) = 1$, where $G(f)$ is the Fourier transform of $g(\tau)$,



determine the transfer function $\lim_{n\to\infty}\prod_{i=1}^{n} H_i(f)$ of the limiting filter $h_1 \star h_2 \star \cdots \star h_n$ as $n$ goes to infinity.

Hint: Determine the relation between $H_i(f)$ and $G(f)$.

(d) (8 pt.) Re-defining in (c) that

$$h_i(\tau) = \left(1 - \frac{\lambda}{n}\right)\delta(\tau) + \frac{\lambda}{n}\delta(\tau - \tau_0) \text{ for } 1 \le i \le n,$$

determine the transfer function of the limiting filter as $n$ goes to infinity, where $\delta()$ is the Dirac delta function.

Hint: Determine the relation between $H_i(f)$ and $G(f)$.

**Solution.**

(a) By the second and fourth cumulants, we know that

$$E[S_n^4] - 3E^2[S_n^2] = n\left(E[X^4] - 3E^2[X^2]\right)$$

and

$$E[S_n^2] = nE[X^2].$$

This implies

$$\begin{aligned}
E[S_n^4] &= 3E^2[S_n^2] + n\left(E[X^4] - 3E^2[X^2]\right) \\
&= 3n^2E^2[X^2] + n\left(E[X^4] - 3E^2[X^2]\right) \\
&= nE[X^4] + (3n^2 - 3n)E^2[X^2] \\
&= 3n^2 - 2n,
\end{aligned}$$

where the last step follows from $E[X^4] = E[X^2] = 1$. Thus,

$$\lim_{n\to\infty} E\left[\left(\frac{S_n}{\sqrt{n}}\right)^4\right] = \lim_{n\to\infty}\left(3 - \frac{2}{n}\right) = 3,$$

and is equal to $E[N^4]$.

(b) For i.i.d. zero-mean and unit-variance random variables,

$$\begin{aligned}
E[S_n^4] &= 3E^2[S_n^2] + n\left(E[X^4] - 3E^2[X^2]\right) \\
&= 3n^2E^2[X^2] + n\left(E[X^4] - 3E^2[X^2]\right) \\
&= nE[X^4] + (3n^2 - 3n)E^2[X^2] \\
&= nE[X^4] + (3n^2 - 3n).
\end{aligned}$$

Thus, if $E[X^4] < \infty$,

$$\lim_{n\to\infty} E\left[\left(\frac{S_n}{\sqrt{n}}\right)^4\right] = \lim_{n\to\infty}\left(\frac{1}{n}E[X^4] + 3 - \frac{3}{n}\right) = 3.$$

However, if $E[X^4] = \infty$, then

$$\lim_{n\to\infty} E\left[\left(\frac{S_n}{\sqrt{n}}\right)^4\right] = \infty \neq 3.$$

Therefore, the answer to the question is "negative."

(c)

$$\begin{aligned}
H_i(f) &= \int_{-\infty}^{\infty} h_i(\tau)e^{-\imath 2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty} \sqrt{n}\cdot g(\sqrt{n}\tau)e^{-\imath 2\pi f\tau}d\tau \\
&= \int_{-\infty}^{\infty} g(s)e^{-\imath 2\pi f(s/\sqrt{n})}ds \quad (s = \sqrt{n}\tau) \\
&= \int_{-\infty}^{\infty} g(s)e^{-\imath 2\pi (f/\sqrt{n})s}ds \\
&= G(f/\sqrt{n})
\end{aligned}$$

Thus the transfer function of the filter bank is equal to

$$\prod_{i=1}^{n} H_i(f) \;=\; G^n\left(\frac{f}{\sqrt{n}}\right).$$

This implies

$$\begin{aligned}
\lim_{n\to\infty} G^n\left(\frac{f}{\sqrt{n}}\right) &= e^{\lim_{n\to\infty} n\log\left(G\left(\frac{f}{\sqrt{n}}\right)\right)} \\
&= e^{\lim_{s\to 0}\frac{\log(G(fs))}{s^2}} \quad (s = 1/\sqrt{n}) \\
&= e^{\lim_{s\to 0}\frac{f\frac{G'(fs)}{G(fs)}}{2s}} \quad (G(0) = 1) \\
&= e^{\lim_{s\to 0}\frac{f^2\frac{G''(fs)G(fs)-[G'(fs)]^2}{G^2(fs)}}{2}} \quad (G'(0) = 0) \\
&= e^{G''(0)\cdot f^2/2} = e^{f^2/2}.
\end{aligned}$$

(d) $H_i(f) = 1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^{-\imath 2\pi f \tau_0}$ implies the transfer function of the filter bank is equal to

$$\left(1 + \frac{\lambda(e^{-\imath 2\pi f \tau_0} - 1)}{n}\right)^n,$$

which implies

$$\lim_{n\to\infty} \left(1 + \frac{\lambda(e^{-\imath 2\pi f \tau_0} - 1)}{n}\right)^n = e^{\lambda(e^{-\imath 2\pi f \tau_0} - 1)}.$$

5.

(a) (8 pt.) Suppose $\{X_n\}_{n=1}^{\infty}$ are i.i.d. sequences with each $X_i \in \{0, 1\}$ and $0 < \Pr[X_i = 1] < 1$. Let $Y_n = X_1 \oplus X_2 \oplus \cdots \oplus X_n$. Is $\{Y_n\}_{n=1}^{\infty}$ $\alpha$-mixing? Justify your answer.

(b) (8 pt.) Give an example that a first-order 2-state Markov sequence (not necessarily stationary, irreducible, aperiodic, etc.) is not $\alpha$-mixing. Explain what condition among stationarity, irreducibility and aperiodicity is violated in your example.

Hint: Construct a first-order 2-state Markov sequence that has a very, very strong dependence between $X_1$ and $X_n$ even for large $n$.

**Solutions.**

(a) $Y_n = Y_{n-1} \oplus X_n$. Thus by

$$\Pr[Y_n = a_n | Y_{n-1} = a_{n-1}] = \Pr[X_n = a_n \oplus a_{n-1}] > 0,$$

$\{Y_n\}_{n=1}^{\infty}$ is a finite-state first-order irreducible aperiodic Markov sequence. Hence, it is $\alpha$-mixing.

(b) Let the transition probability matrix of the Markov process be

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Assume the initial probability is $\Pr[X_1 = 0] = \Pr[X_1 = 1] = \frac{1}{2}$. Then

$$\begin{aligned} & |\Pr[X_1 = 1 \text{ and } X_n = 1] - \Pr[X_1 = 1]\Pr[X_n = 1]| \\ &= |\Pr[X_1 = 1]\Pr[X_n = 1|X_1 = 1] - \Pr[X_1 = 1]\Pr[X_n = 1]| \\ &= \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot\frac{1}{2} \\ &= \frac{1}{4}. \end{aligned}$$

Hence, this is not an $\alpha$-mixing process.

Since

$$p_{ij}^{(n)} = \Pr[X_n = j|X_1 = i] = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$\{X_n\}_{n=1}^{\infty}$ is not irreducible even though it is aperiodic.

6. (a) (8 pt.) Suppose that $\mu$ is a point mass at the origin, and $\mu\{0\} = \sigma^2$. Let the characteristic function of random variable $X$ be

$$\varphi_X(t) = \exp\left\{\int_{\Re}\left(e^{itx} - 1\right)\frac{1}{x}\mu(dx)\right\}.$$

Determine the distribution of $X$.

(b) (8 pt.) Re-do (a) when $\mu$ consists of a point mass $\lambda x$ at some $x \neq 0$.

Hint: The characteristic function of a Poisson random variable with parameter $\lambda$ is $E[e^{\imath tX}] = \exp\{\lambda(e^{\imath t} - 1)\}$.

(c) (10 pt.) If an independent triangular array $\{X_{n,k}\}_{k=1}^{r_n}$ satisfies

i. $\lim_{n\to\infty}\max_{1\le k\le r_n} E[X_{n,k}^2] = 0$;

ii. $\sup_{n\ge 1} s_n^2 < \infty$, where $s_n^2 = \sum_{k=1}^{r_n} E[X_{n,k}^2]$,

prove that

$$\lim_{n\to\infty}\max_{1\le k\le r_n}\left|\varphi_{X_{n,k}}(t) - \exp\left\{\varphi_{X_{n,k}}(t) - 1\right\}\right| = 0,$$

where $\varphi_{X_{n,k}}(t) = E[e^{\imath tX_{n,k}}]$ is the characteristic function of $X_{n,k}$.

Hint: See Slide 28-19.

**Solution.**

(a)

$$\begin{aligned}
\varphi_X(t) &= \exp\left\{\int_{\Re}\left(e^{itx}-1\right)\frac{1}{x}\mu(dx)\right\} \\
&= \exp\left\{\sigma^2\lim_{x\to 0}\frac{\left(e^{itx}-1\right)}{x}\right\} \\
&= \exp\left\{\sigma^2\lim_{x\to 0}\frac{\left(ite^{itx}\right)}{1}\right\} \\
&= \exp\left\{\imath t\sigma^2\right\}.
\end{aligned}$$

Thus $X$ is deterministic with distribution $\Pr[X=\sigma^2]=1$.

(b)

$$\begin{aligned}
\varphi_X(t) &= \exp\left\{\int_{\Re}\left(e^{itx}-1\right)\frac{1}{x}\mu(dx)\right\} \\
&= \exp\left\{\lambda\left(e^{itx}-1\right)\right\},
\end{aligned}$$

Hence, $X$ has the same distribution as $xZ_\lambda$, where $Z_\lambda$ has Poisson distribution with mean $\lambda$.

(c)

$$\begin{aligned}
&\max_{1\le k\le r_n}\left|\varphi_{X_{n,k}}(t)-\exp\left\{\varphi_{X_{n,k}}(t)-1\right\}\right| \\
&\le \sum_{k=1}^{r_n}\left|\varphi_{X_{n,k}}(t)-\exp\left\{\varphi_{X_{n,k}}(t)-1\right\}\right| \\
&= \sum_{k=1}^{r_n}\left|1+\theta_{n,k}(t)-e^{\theta_{n,k}(t)}\right| \\
&\le \sum_{k=1}^{r_n}|\theta_{n,k}(t)|^2 e^{|\theta_{n,k}(t)|} \\
&\le e^{t^2s_n^2/2}\sum_{k=1}^{r_n}|\theta_{n,k}(t)|^2 \\
&\le e^{t^2s_n^2/2}\left(\max_{1\le k\le r_n}|\theta_{n,k}(t)|\right)\sum_{k=1}^{r_n}|\theta_{n,k}(t)| \\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$