2016 Final for Advanced Probability for Communications

The number of points is 116 in this exam. Good luck.

1. (8 pt.) The Law of the Iterated Logarithm states that for i.i.d. $\{X_i\}_{i=1}^{\infty}$ with mean 0 and variance 1,

$$\Pr\left[\limsup_{n\to\infty}\frac{S_n}{\sqrt{2n\log\log(n)}}=1\right]=1$$

where $S_n = X_1 + \cdots + X_n$. Then for arbitrarily small $\epsilon > 0$, what is the limiting probability for

$$\limsup_{n\to\infty}\Pr\left[\left|\frac{S_n}{\sqrt{2n\log\log(n)}}-1\right|>\epsilon\right]?$$

Justify your answer.

Hint: $S_n/\sqrt{n} \Rightarrow N$, where N is standard normal distributed.

2. Define a cdf F recursively via the following four rules.

   1) $F(0) = 0$ and $F(1) = 1$;
   2) $F(x) = \frac{1}{2}$ for $\frac{1}{3} < x < \frac{2}{3}$;
   3) $F(x/3) = \frac{1}{2}F(x)$ for $0 \le x \le 1$;
   4) $F(1-x) = 1 - F(x)$.

   By the rules, we obtain

$$F\left(\tfrac{1}{3}\right)=\tfrac{1}{2}F(1)=\tfrac{1}{2};\quad F\left(\tfrac{2}{3}\right)=1-F\left(\tfrac{1}{3}\right)=\tfrac{1}{2},$$

   and

$$\begin{array}{ll} F\left(\frac{1}{9}\right)=\frac{1}{2}F\left(\frac{1}{3}\right)=\frac{1}{4}; & F\left(\frac{2}{9}\right)=\frac{1}{2}F\left(\frac{2}{3}\right)=\frac{1}{4}; \\ F\left(\frac{3}{9}\right)=F\left(\frac{1}{3}\right)=\frac{1}{2}; & F\left(\frac{4}{9}\right)=\frac{1}{2}; \\ F\left(\frac{5}{9}\right)=1-F\left(\frac{4}{9}\right)=\frac{1}{2}; & F\left(\frac{6}{9}\right)=1-F\left(\frac{3}{9}\right)=\frac{1}{2}; \\ F\left(\frac{7}{9}\right)=1-F\left(\frac{2}{9}\right)=\frac{3}{4}; & F\left(\frac{8}{9}\right)=1-F\left(\frac{1}{9}\right)=\frac{3}{4}. \end{array}$$

   We can continue the above procedure to obtain the F-function values for $x = k/3^{\ell}$ for every $\ell \ge 3$ and $0 \le k \le 3^{\ell}$.

Based on the setting, the Riemann upper and lower approximations of $\int_0^1 x\,dF(x)$ with $\Delta x = 3^{-2}$ is given by:

$$\begin{aligned}
L_2 &\triangleq \sum_{i=0}^{8} \left(\frac{i}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] \\
&\leq \int_0^1 x\,dF(x) \leq \sum_{i=0}^{8} \left(\frac{i+1}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] \triangleq U_2,
\end{aligned}$$

where

$$L_2 = \sum_{i=0}^{8} \left(\frac{i}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] = \frac{(0+2+6+8)}{9}\cdot\frac{1}{4} = \frac{4}{9}$$

and

$$U_2 = \sum_{i=0}^{8} \left(\frac{i+1}{9}\right)\left[F\left(\frac{i+1}{9}\right) - F\left(\frac{i}{9}\right)\right] = \frac{(1+3+7+9)}{9}\cdot\frac{1}{4} = \frac{5}{9}.$$

The Riemann upper and lower approximations of $\int_0^1 x\,dF(x)$ with $\Delta x = 3^{-3}$ is given by

$$\begin{aligned}
L_3 &\triangleq \sum_{i=0}^{26} \left(\frac{i}{27}\right)\left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \\
&\leq \int_0^1 x\,dF(x) \leq \sum_{i=0}^{26} \left(\frac{i+1}{27}\right)\left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \triangleq U_3,
\end{aligned}$$

where

$$\begin{aligned}
L_3 &= \sum_{i=0}^{26} \left(\frac{i}{27}\right)\left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \\
&= \frac{(0+2+6+8)+(18+20+24+26)}{27}\cdot\frac{1}{8} = \frac{13}{27}
\end{aligned}$$

and

$$\begin{aligned}
U_3 &= \sum_{i=0}^{26} \left(\frac{i+1}{27}\right)\left[F\left(\frac{i+1}{27}\right) - F\left(\frac{i}{27}\right)\right] \\
&= \frac{(1+3+7+9)+(19+21+25+27)}{27}\cdot\frac{1}{8} = \frac{14}{27}.
\end{aligned}$$

Noting that $F\left(\frac{i+1}{3^\ell}\right) - F\left(\frac{i}{3^\ell}\right)$ is either 0 or $2^{-\ell}$, answer the following questions.

(a) (8 pt.) Find the Riemann approximation of $\int_0^1 x\,dF(x)$ with $\Delta x = 3^{-\ell}$. In other words, determine the upper and lower bounds of the below equation:

$$\begin{aligned} L_\ell &\triangleq \sum_{i=0}^{3^\ell - 1} \left(\frac{i}{3^\ell}\right)\left[F\left(\frac{i+1}{3^\ell}\right) - F\left(\frac{i}{3^\ell}\right)\right] \\ &\le \int_0^1 x\,dF(x) \le \sum_{i=0}^{8} \left(\frac{i+1}{3^\ell}\right)\left[F\left(\frac{i+1}{3^\ell}\right) - F\left(\frac{i}{3^\ell}\right)\right] \triangleq U_\ell. \end{aligned}$$

Hint: Based on L_2 and L_3, deduce the formula of L_ℓ. Then deduce the relation between L_ℓ and U_ℓ.

(b) (8 pt.) Determine $\lim_{\ell\to\infty} L_\ell$ and $\lim_{\ell\to\infty} U_\ell$ and check whether they are equal or not.

3. (a) (8 pt.) Let $\{P_i\}_{i=1}^m$ and $\{Q_i\}_{i=1}^m$ be positive real numbers satisfying

$$\sum_{i=1}^m P_i = \sum_{i=1}^m Q_i = 1.$$

Prove that for $0 < \lambda < 1$,

$$\sum_{i=1}^m Q_i^\lambda P_i^{1-\lambda} \le 1,$$

and give a necessary and sufficient condition for equality.

Hint: Note that $\sum_{i=1}^m Q_i^\lambda P_i^{1-\lambda} = \sum_{i=1}^m P_i \left(\frac{Q_i}{P_i}\right)^\lambda = E[f(X)]$, where $f(x) = x^\lambda$ and X is a random variable with alphabet

$$\left\{\frac{Q_1}{P_1}, \frac{Q_2}{P_2}, \ldots, \frac{Q_m}{P_m}\right\}$$

and distribution $\Pr[X = Q_i/P_i] = P_i$. Then apply Jensen's inequality on $E[f(X)]$.

(b) (8 pt.) Continue from (a). If $\lambda > 1$ or $\lambda < 0$, does

$$\sum_{i=1}^{m} Q_i^{\lambda} P_i^{1-\lambda} \le 1$$

still hold ? Justify your answer.

(c) (10 pt.) (General Hölder's inequality) Prove that for positive real numbers $\{a_i\}_{i=1}^m$ and $\{b_i\}_{i=1}^m$,

$$\sum_{i=1}^{m} a_i b_i \begin{cases} \le \left(\sum_{i=1}^m a_i^{1/\lambda}\right)^{\lambda} \left(\sum_{i=1}^m b_i^{1/(1-\lambda)}\right)^{1-\lambda}, & 0 < \lambda < 1; \\ \ge \left(\sum_{i=1}^m a_i^{1/\lambda}\right)^{\lambda} \left(\sum_{i=1}^m b_i^{1/(1-\lambda)}\right)^{1-\lambda}, & \lambda < 0 \text{ or } \lambda > 1; \end{cases}$$

with equality holding if, and only if, for some constant c, $a_i^{1/\lambda} = c b_i^{1/(1-\lambda)}$ for all i.

Hint: Use (a) and (b) with properly setting P_i and Q_i. For example, you may set $P_i = \frac{b_i^{1/\lambda}}{\sum_{i=1}^m b_i^{1/\lambda}}$.

4. (a) (8 pt.) Suppose that $\{X_n\}_{n=1}^{\infty}$ is an independent sequence of random variables having the same distribution $\Pr[X_i = -1] = \Pr[X_i = 1] = \frac{1}{2}$. Then Theorem 27.1 states that

$$\frac{S_n}{\sqrt{n}} \Rightarrow N,$$

where N is standard normal distributed, and $S_n = X_1 + \cdots + X_n$. Find the fourth moment of $S_n/\sqrt{n}$ using cumulant formulas. Is it asymptotically equal to $E[N^4]$ as n goes to infinity?

Hint: The cumulants of a zero-mean random variable X are given by

$$\begin{aligned} C^{(1)}(0) = c_1 &= 0 \\ C^{(2)}(0) = c_2 &= E[X^2] = \mathrm{Var}[X] \\ C^{(3)}(0) = c_3 &= E[X^3] \\ C^{(4)}(0) = c_4 &= E[X^4] - 3E^2[X^2] \\ &\vdots \end{aligned}$$

Recall why these terms are named "cumulants." Also, $E[N^2] = 1$ and $E[N^4] = 3$.

(b) (8 pt.) Is it always true that for i.i.d. zero-mean and unit variance random variables $\{X_i\}_{i=1}^{\infty}$,

$$\lim_{n\to\infty} E[S_n^4/n^2] = E[N^4] = 3.$$

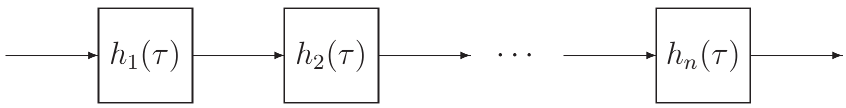
If your answer is affirmative, prove it. If not, justify your answer.

Hint: Do we assume that X_i has finite fourth moment?

(c) (8 pt.) As shown in the figure below, for the tandem filtering bank with

$$h_1(\tau) = h_2(\tau) = \cdots = h_n(\tau) = \sqrt{n} \cdot g(\sqrt{n}\tau)$$

satisfying $G(0) = 1$, $G'(0) = 0$ and $|G''(0)| < \infty$, where $G(f)$ is the Fourier transform of $g(\tau)$,



determine the transfer function $\lim_{n\to\infty} \prod_{i=1}^{n} H_i(f)$ of the limiting filter $h_1 \star h_2 \star \cdots \star h_n$ as n goes to infinity.

Hint: Determine the relation between $H_i(f)$ and $G(f)$.

(d) (8 pt.) Re-defining in (c) that

$$h_i(\tau) = \left(1 - \frac{\lambda}{n}\right)\delta(\tau) + \frac{\lambda}{n}\delta(\tau - \tau_0) \text{ for } 1 \le i \le n,$$

determine the transfer function of the limiting filter as n goes to infinity, where $\delta()$ is the Dirac delta function.

Hint: Determine the relation between $H_i(f)$ and $G(f)$.

5.

(a) (8 pt.) Suppose $\{X_n\}_{n=1}^{\infty}$ are i.i.d. sequences with each $X_i \in \{0, 1\}$ and $0 < \Pr[X_i = 1] < 1$. Let $Y_n = X_1 \oplus X_2 \oplus \cdots \oplus X_n$. Is $\{Y_n\}_{n=1}^{\infty}$ α-mixing? Justify your answer.

(b) (8 pt.) Give an example that a first-order 2-state Markov sequence (not necessarily stationary, irreducible, aperiodic, etc.) is not α-mixing. Explain what condition among stationarity, irreducibility and aperiodicity is violated in your example.

Hint: Construct a first-order 2-state Markov sequence that has a very, very strong dependence between X_1 and X_n even for large n.

6. (a) (8 pt.) Suppose that μ is a point mass at the origin, and $\mu\{0\} = \sigma^2$. Let the characteristic function of random variable X be

$$\varphi_X(t) = \exp\left\{ \int_{\Re} \left(e^{itx} - 1\right) \frac{1}{x} \mu(dx) \right\}.$$

Determine the distribution of X.

(b) (8 pt.) Re-do (a) when μ consists of a point mass λx at some $x \neq 0$.

Hint: The characteristic function of a Poisson random variable with parameter λ is $E[e^{\imath tX}] = \exp\{\lambda(e^{\imath t} - 1)\}$.

(c) (10 pt.) If an independent triangular array $\{X_{n,k}\}_{k=1}^{r_n}$ satisfies

i. $\lim_{n\to\infty} \max_{1\le k\le r_n} E[X_{n,k}^2] = 0$;

ii. $\sup_{n\ge 1} s_n^2 < \infty$, where $s_n^2 = \sum_{k=1}^{r_n} E[X_{n,k}^2]$,

prove that

$$\lim_{n\to\infty} \max_{1\le k\le r_n} \left|\varphi_{X_{n,k}}(t) - \exp\left\{\varphi_{X_{n,k}}(t) - 1\right\}\right| = 0,$$

where $\varphi_{X_{n,k}}(t) = E[e^{\imath tX_{n,k}}]$ is the characteristic function of $X_{n,k}$.

Hint: See Slide 28-19.