Appendix B: Error Probability for Multichannel Binary Signals

Po-Ning Chen, Professor

Institute of Communications Engineering

National Chiao Tung University

Hsin Chu, Taiwan 30010, R.O.C.

General quadratic form

Determine $\Pr\{D < 0\}$, where

$$D = \sum_{k=1}^{L} \left(A|X_k|^2 + B|Y_k|^2 + C X_k Y_k^* + C^* X_k^* Y_k \right),$$

A, B and C are complex constants satisfying $|C|^2 - AB > 0$, X_k and Y_k are a pair of correlated complex-valued Gaussian random variable, and $\{X_k, Y_k\}_{k=1}^{L}$ are mutually statistically independent with common covariance matrix.

- This can be determined by means of the characteristic function technique.
- Specifically, let $f_D(d)$ be the pdf of D, and let $\varphi_D(v) = \int_{-\infty}^{\infty} f_D(x) e^{ivx} dx$ be the characteristic function of D. Then, Uniqueness Theorem (Theorem 26.2) tells that:

$$\Pr\{D < 0\} = \lim_{T\to\infty} -\frac{1}{2\pi i} \int_{-T}^{T} \frac{\varphi_D(v)}{v} dv.$$

General quadratic form

- Let $D_k = A|X_k|^2 + B|Y_k|^2 + CX_kY_k^* + C^*X_k^*Y_k$. Then, the characteristic function of D_k is:

$$\varphi_{D_k}(v) = \frac{v_1 v_2}{(v + iv_1)(v - iv_2)} \exp\left[-\frac{v_1 v_2 (v^2 \alpha_{1k} - iv\alpha_{2k})}{(v + iv_1)(v - iv_2)}\right],$$

where

$$v_1 = \sqrt{w^2 + \frac{1}{4(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2)(|C|^2 - AB)}} - w$$

$$v_2 = \sqrt{w^2 + \frac{1}{4(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2)(|C|^2 - AB)}} + w$$

$$w = \frac{A\mu_{xx} + B\mu_{yy} + C\mu_{xy}^* + C^*\mu_{xy}}{4(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2)(|C|^2 - AB)}$$

General quadratic form

$$
\begin{aligned}
\mu_{xx} &= \frac{1}{2}E[(X_k - m_{k,x})(X_k - m_{k,x})^*] \\
\mu_{yy} &= \frac{1}{2}E[(Y_k - m_{k,y})(Y_k - m_{k,y})^*] \\
\mu_{xy} &= \frac{1}{2}E[(X_k - m_{k,x})(Y_k - m_{k,y})^*] \\
m_{k,x} &= E[X_k] \\
m_{k,y} &= E[Y_k] \\
\alpha_{1k} &= 2(|C|^2 - AB)(|m_{k,x}|^2\mu_{yy} + |m_{k,y}|^2\mu_{xx} - m_{k,x}^* m_{k,y}\mu_{xy} - m_{k,x}m_{k,y}^*\mu_{xy}^*) \\
\alpha_{2k} &= A|m_{k,x}|^2 + B|m_{k,y}|^2 + Cm_{k,x}^* m_{k,y} + C^* m_{k,x}m_{k,y}^*
\end{aligned}
$$

- As $\{D_k\}_{k=1}^L$ are assumed independent,

$$
\varphi_D(v) = \prod_{k=1}^{L} \varphi_{D_k}(v) = \frac{(v_1 v_2)^L}{(v + iv_1)^L (v - iv_2)^L} \exp\left[-\frac{v_1 v_2(v^2\alpha_1 - iv\alpha_2)}{(v + iv_1)(v - iv_2)}\right],
$$

where $\alpha_1 = \sum_{k=1}^L \alpha_{1k}$ and $\alpha_2 = \sum_{k=1}^L \alpha_{2k}$.

General quadratic form

- Takeing $\varphi_D(v)$ into $\Pr\{D<0\}$, we obtain:

$$\begin{aligned}\Pr\{D<0\} &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\varphi_D(v)}{v}dv \\ &= -\frac{(v_1v_2)^L}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{v(v+iv_1)^L(v-iv_2)^L}\exp\left[-\frac{v_1v_2(v^2\alpha_1-iv\alpha_2)}{(v+iv_1)(v-iv_2)}\right]dv \\ &= -\frac{(v_1v_2)^L}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{v(v+iv_1)^L(v-iv_2)^L}\exp\left[-A_1+\frac{iA_2}{v+iv_1}-\frac{iA_3}{v-iv_2}\right]dv,\end{aligned}$$

where

$$\begin{aligned}A_1 &= \alpha_1v_1v_2 \\ A_2 &= \frac{v_1^2v_2}{v_1+v_2}(\alpha_1v_1+\alpha_2) \\ A_3 &= \frac{v_1v_2^2}{v_1+v_2}(\alpha_1v_2-\alpha_2).\end{aligned}$$

General quadratic form

- By taking a conformal transformation from v plane to p plane through

$$p = -\frac{v_1}{v_2}\left(\frac{v - iv_2}{v + iv_1}\right),$$

the above derivation continues:

$$\Pr\{D < 0\} = \frac{1}{(1 + v_2/v_1)^{2L-1}} \exp\left[-\frac{1}{2}(a^2 + b^2)\right] \frac{1}{2\pi i} \oint_{\Gamma} f(p)dp,$$

where

$$f(p) = \frac{[1 + (v_2/v_1)p]^{2L-1}}{p^L(1 - p)} \exp\left[\frac{b^2}{2}p + \frac{a^2}{2}\frac{1}{p}\right]$$

$a = \sqrt{2A_3(v_1/v_2)/(v_1 + v_2)}$, $b = \sqrt{2A_2(v_2/v_1)/(v_1 + v_2)}$, and Γ is a circular contour of radius less than unity that encloses the origin.

General quadratic form

- Next, binomial-expanding $[1 + (v_2/v_1)p]^{2L-1}$, we evaluate

$$
\begin{aligned}
\frac{1}{2\pi i}\oint_\Gamma f(p)dp &= \sum_{k=0}^{2L-1} \binom{2L-1}{k} \left(\frac{v_2}{v_1}\right)^k \times \frac{1}{2\pi i}\oint_\Gamma \frac{p^k}{p^L(1-p)} \exp\left[\frac{b^2}{2}p + \frac{a^2}{2}\frac{1}{p}\right] dp \\
&= \sum_{k=0}^{2L-1} \binom{2L-1}{k} \left(\frac{v_2}{v_1}\right)^k \times \frac{1}{2\pi i}\oint_\Gamma \frac{1}{p^{L-k}(1-p)} \exp\left[\frac{b^2}{2}p + \frac{a^2}{2}\frac{1}{p}\right] dp
\end{aligned}
$$

General quadratic form

$$I_n(ab) = \begin{cases} \dfrac{1}{2\pi i}\left(\dfrac{a}{b}\right)^n \oint_\Gamma \dfrac{1}{p^{n+1}} \exp\left[\dfrac{b^2}{2}p + \dfrac{a^2}{2}\dfrac{1}{p}\right] dp \\ \dfrac{1}{2\pi i}\left(\dfrac{b}{a}\right)^n \oint_\Gamma p^{n-1} \exp\left[\dfrac{b^2}{2}p + \dfrac{a^2}{2}\dfrac{1}{p}\right] dp \end{cases}$$

and

$$\begin{aligned} Q_m(a,b) &= \exp\left[-\frac{1}{2}(a^2+b^2)\right] \frac{1}{2\pi i} \oint_\Gamma \frac{1}{p^m(1-p)} \exp\left[\frac{b^2}{2}p + \frac{a^2}{2}\frac{1}{p}\right] dp \\ &= \int_b^\infty x(x/a)^{m-1} \exp\left[-\frac{1}{2}(x^2+a^2)\right] I_{m-1}(ax)dx, \end{aligned}$$

which results in $Q_1(a,b) = \exp\left[-\frac{1}{2}(a^2+b^2)\right] \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n I_n(ab)$, where $I_n(x)$ is the nth-order modified Bessel function of the first kind, and $Q_m(a,b)$ is the generalized Marcum's Q function.

General quadratic form

- Finally,

$$\frac{1}{2\pi i}\oint_\Gamma \frac{1}{p^{L-k}(1-p)} \exp\left[\frac{b^2}{2}p + \frac{a^2}{2}\frac{1}{p}\right] dp$$

$$= \begin{cases} Q_1(a,b)\exp\left[\frac{1}{2}(a^2+b^2)\right] + \displaystyle\sum_{n=1}^{L-1-k}\left(\frac{b}{a}\right)^n I_n(ab), & 0 \le k \le L-2 \\ Q_1(a,b)\exp\left[\frac{1}{2}(a^2+b^2)\right], & k = L-1 \\ Q_1(a,b)\exp\left[\frac{1}{2}(a^2+b^2)\right] - \displaystyle\sum_{n=0}^{k-L}\left(\frac{a}{b}\right)^n I_n(ab), & L \le k \le 2L-1. \end{cases}$$

General quadratic form

and for $L > 1$,

$$\Pr\{D < 0\} = Q_1(a,b) - I_0(ab)\exp\left[-\frac{1}{2}(a^2+b^2)\right] + \frac{I_0(ab)\exp\left[-\frac{1}{2}(a^2+b^2)\right]}{(1+v_2/v_1)^{2L-1}}\sum_{k=0}^{L-1}\binom{2L-1}{k}\left(\frac{v_2}{v_1}\right)^k$$

$$+\frac{\exp\left[-\frac{1}{2}(a^2+b^2)\right]}{(1+v_2/v_1)^{2L-1}}\sum_{n=1}^{L-1} I_n(ab)\sum_{k=0}^{L-1-n}\binom{2L-1}{k}\left[\left(\frac{b}{a}\right)^n\left(\frac{v_2}{v_1}\right)^k - \left(\frac{a}{b}\right)^n\left(\frac{v_2}{v_1}\right)^{2L-1-k}\right]$$

and for $L = 1$,

$$\Pr\{D < 0\} = Q_1(a,b) - \frac{v_2/v_1}{1+v_2/v_1} I_0(ab)\exp\left[-\frac{1}{2}(a^2+b^2)\right],$$

where

$$a = \left[\frac{2v_1^2 v_2(\alpha_1 v_2 - \alpha_2)}{(v_1+v_2)^2}\right]^{1/2} \quad \text{and} \quad b = \left[\frac{2v_1 v_2^2(\alpha_1 v_2 + \alpha_2)}{(v_1+v_2)^2}\right]^{1/2}.$$

Example: Probability of Error for Differential PSK

$$\text{DBPSK} = \begin{cases} \text{Shift the phase w.r.t. the previous symbol by 0 degree,} & \text{if Input} = 0 \\ \text{Shift the phase w.r.t. the previous symbol by 180 degree,} & \text{if Input} = 1 \end{cases}$$

$$\text{DQPSK} = \begin{cases} \text{Shift the phase w.r.t. the previous symbol by 0 degree,} & \text{if Input} = 00 \\ \text{Shift the phase w.r.t. the previous symbol by 90 degree,} & \text{if Input} = 01 \\ \text{Shift the phase w.r.t. the previous symbol by 180 degree,} & \text{if Input} = 11 \\ \text{Shift the phase w.r.t. the previous symbol by 270 degree,} & \text{if Input} = 10 \end{cases}$$

At time k, the complex reception is given by:

$$r_k = \sqrt{E}\exp\{i(\theta_k - \phi)\} + n_k,$$

where ϕ is a constant offset, and n_k is zero-mean independent complex Gaussian distributed with marginal covariance $N_0/2$. The demodulation is based on

$$\begin{aligned} r_k r_{k-1}^* &= (\sqrt{E}\exp\{i(\theta_k - \phi)\} + n_k)(\sqrt{E}\exp\{i(\theta_{k-1} - \phi)\} + n_{k-1})^* \\ &= E\exp\{i(\theta_k - \theta_{k-1})\} + \left[\sqrt{E}\exp\{i(\theta_k - \phi)\}n_{k-1}^* + \sqrt{E}\exp\{i(\theta_{k-1} - \phi)\}n_k + n_k n_{k-1}^*\right]. \end{aligned}$$

It seems hard to find the exact error probability formula since $n_k n_{k-1}^*$ is no longer Gaussian!!

Example: Probability of Error for Differential PSK

For DBPSK, $\begin{cases} \theta_k - \theta_{k-1} = 0 \\ \theta_k - \theta_{k-1} = \pi \end{cases}$ implies $\begin{cases} I_k = 0 \\ I_k = 1 \end{cases}$.

So based on uniform prior, the optimal decision rule should be:

$$\begin{cases} \mathrm{Re}\{r_k r_{k-1}^*\} > 0 \\ \mathrm{Re}\{r_k r_{k-1}^*\} < 0 \end{cases} \text{ implies } \begin{cases} I_k = 0 \\ I_k = 1 \end{cases}$$

As a result, letting

$$\begin{cases} X_1 = r_k, \text{ a complex Gaussian} \\ Y_1 = r_{k-1}, \text{ a complex Gaussian} \\ L = 1 \\ A = B = 0 \\ C = 1 \end{cases}$$

we have $D = X_1 Y_1^* + X_1^* Y_1 = 2\mathrm{Re}\{X_1 Y_1^*\} = 2\mathrm{Re}\{r_k r_{k-1}^*\}$.

Example: Probability of Error for Differential PSK

Under $(\theta_k - \theta_{k-1}) = 0$, we have $P_\mathrm{b} = \Pr\{D < 0\}$ with

$$\begin{aligned}
m_{1,x} &= E[X_1] = E[r_k] = \sqrt{E}\exp\{i(\theta_k - \phi)\} \\
m_{1,y} &= E[Y_1] = E[r_{k-1}] = \sqrt{E}\exp\{i(\theta_{k-1} - \phi)\} \\
\mu_{xx} &= \frac{1}{2}E[(X_1 - m_{1,x})(X_1 - m_{1,x})^*] = \frac{1}{2}E[n_k n_k^*] = N_0/2 \\
\mu_{yy} &= \frac{1}{2}E[(Y_1 - m_{1,y})(Y_1 - m_{1,y})^*] = \frac{1}{2}E[n_{k-1} n_{k-1}^*] = N_0/2 \\
\mu_{xy} &= \frac{1}{2}E[(X_1 - m_{1,x})(Y_1 - m_{1,y})^*] = \frac{1}{2}E[n_k n_{k-1}^*] = \frac{1}{2}E[n_k]E[n_{k-1}^*] = 0 \\
\alpha_{1,1} &= 2(|C|^2 - AB)(|m_{1,x}|^2\mu_{yy} + |m_{1,y}|^2\mu_{xx} - m_{1,x}^* m_{1,y}\mu_{xy} - m_{1,x}m_{1,y}^*\mu_{xy}^*) \\
&= 2(EN_0/2 + EN_0/2 - 0 - 0) = 2EN_0 \\
\alpha_{2,1} &= A|m_{1,x}|^2 + B|m_{1,y}|^2 + Cm_{1,x}^* m_{1,y} + C^* m_{1,x}m_{1,y}^* \\
&= 0 + 0 + E\exp\{i(\theta_{k-1} - \theta_k)\} + E\exp\{i(\theta_k - \theta_{k-1})\} = 2E
\end{aligned}$$

Example: Probability of Error for Differential PSK

$$
\begin{aligned}
w &= \frac{A\mu_{xx} + B\mu_{yy} + C\mu_{xy}^* + C^*\mu_{xy}}{4(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2)(|C|^2 - AB)} = \frac{0+0+0+0}{4(N_0^2/4 - 0)} = 0 \\
v_1 &= \sqrt{w^2 + \frac{1}{4(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2)(|C|^2 - AB)}} - w = \sqrt{0 + \frac{1}{4(N_0^2/4 - 0)}} - 0 = 1/N_0 \\
v_2 &= \sqrt{w^2 + \frac{1}{4(\mu_{xx}\mu_{yy} - |\mu_{xy}|^2)(|C|^2 - AB)}} + w = \sqrt{0 + \frac{1}{4(N_0^2/4 - 0)}} + 0 = 1/N_0 \\
a &= \left[\frac{2v_1^2 v_2(\alpha_1 v_2 - \alpha_2)}{(v_1 + v_2)^2}\right]^{1/2} = \left[\frac{2(1/N_0)^2(1/N_0)(2EN_0/N_0 - 2E)}{(1/N_0 + 1/N_0)^2}\right]^{1/2} = 0 \\
b &= \left[\frac{2v_1 v_2^2(\alpha_1 v_2 + \alpha_2)}{(v_1 + v_2)^2}\right]^{1/2} = \left[\frac{2(1/N_0)(1/N_0)^2(2EN_0/N_0 + 2E)}{(1/N_0 + 1/N_0)^2}\right]^{1/2} = \sqrt{2E/N_0} = \sqrt{2\gamma}
\end{aligned}
$$

Example: Probability of Error for Differential PSK

These lead to:

$$\begin{aligned} P_{\mathrm{b}} &= \Pr\{D < 0\} \\ &= Q_1(a,b) - \frac{v_2/v_1}{1+v_2/v_1} I_0(ab)\exp\left[-\frac{1}{2}(a^2+b^2)\right] \\ &= Q_1(0,\sqrt{2\gamma}) - \frac{1}{2} I_0(0)\exp\left[-\gamma\right] \\ &= \frac{1}{2}\exp\left(-\gamma\right) \end{aligned}$$

$$\lim_{x\downarrow 0} I_n(x) = \begin{cases} 1, & n = 0 \\ 0, & n \geq 1 \end{cases}$$

and

$$\lim_{a\downarrow 0} Q_1(a,b) = \lim_{a\downarrow 0} \exp\left[-\frac{1}{2}(a^2+b^2)\right] \sum_{n=0}^{\infty} \left(\frac{a}{b}\right)^n I_n(ab) = \exp\left[-\frac{1}{2}b^2\right].$$