

Berry-Esseen Theorem

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Historical aspects

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- The central limit theorem (CLT) concerns the situation that the limit distribution of the normalized sum is normal.
- As an example, for i.i.d. zero-mean sequence X_1, X_2, \dots ,

$$\frac{X_1 + \dots + X_n}{\sqrt{nE[X_1^2]}} \Rightarrow N,$$

where N has standard normal distribution.

- **Question:** What is the rate of convergence of normalized sum distribution to standard normal distribution?

Historical aspects

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- The first convergence rate estimates in the CLT were obtained by A. M. Lyapounov in 1900-1901.
- In the beginning of 1940s, the classic Berry-Esseen estimate came to the light:

$$\sup_{x \in \mathfrak{R}} |F_n(x) - \Phi(x)| \leq C \frac{\beta_3}{\sigma^3 \sqrt{n}},$$

where

- F_n is the cdf of $(X_1 + \cdots + X_n) / \sqrt{nE[X_1^2]}$,
 - Φ is the standard normal cdf,
 - $\beta_3 = E \left[|X - E[X]|^3 \right]$,
 - $\sigma^2 = E \left[|X - E[X]|^2 \right]$, and
 - C is a universal constant, independent of n .
- In fact, it was due to Lyapounov that the finiteness of $(2 + \delta)$ th absolute moment, where $0 < \delta < 1$, guarantees

$$\sup_{x \in \mathfrak{R}} |F_n(x) - \Phi(x)| = O(n^{-\delta/2}) \text{ as } n \rightarrow \infty.$$

Berry-Esseen theorem

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- Classic Berry-Esseen theorem

- For independent zero-mean random variables $\{X_i\}_{i=1}^n$,

$$\sup_{a \in \mathfrak{R}} \left| \Pr \left[\frac{1}{s_n} (X_1 + \dots + X_n) \leq a \right] - \Phi(a) \right| \leq C \frac{r_n}{s_n^3},$$

where

- * $s_n^2 \equiv$ sum of the marginal variances,

- * $r_n \equiv$ sum of the marginal absolute 3rd central moments,

- * $C \equiv$ absolute constant, and

- * $\Phi(\cdot) \equiv$ the unit Gaussian cumulative distribution function (cdf).

- Notably, only the *first three moments* are involved in the Berry-Esseen inequality.

Berry-Esseen theorem

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$$\sup_{a \in \mathbb{R}} \left| \Pr \left[\frac{1}{s_n} (X_1 + \dots + X_n) \leq a \right] - \Phi(a) \right| \leq C \frac{r_n}{s_n^3}.$$

- (Feller's book, 1979)
 - $C = 6$ for an independent sample sum.
 - $C = 3$ for an independent and identically distributed sample sum.
- (Beek 1972) $C = 0.7975$ for an independent sample sum.
- (Shiganov 1986)
 - $C = 0.7915$ for an independent sample sum.
 - $C = 0.7655$ for an independent and identically distributed sample sum.

Berry-Esseen theorem

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- Again, the remarkable aspect of the Berry-Esseen theorem is that the upper bound depends only on the variance and the 3rd central moment.
- Hence, it can provide a good probability estimate based on the first three moments.

Berry-Esseen theorem and probability bound

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Technique (behind the proof): Pass the difference through a bandlimited filter.

Lemma Fix a symmetric bandlimited filtering function

$$v_T(x) = \frac{1 - \cos(Tx)}{\pi T x^2} = \frac{2 \sin^2(Tx/2)}{\pi T x^2}$$

with characteristic function

$$\omega_T(\zeta) \triangleq \int_{-\infty}^{\infty} v_T(x) e^{-j\zeta x} dx = \begin{cases} 1 - \frac{|\zeta|}{T}, & \text{if } |\zeta| \leq T; \\ 0, & \text{otherwise.} \end{cases}$$

For any cdf $H(\cdot)$ on the real line \Re ,

$$\sup_{x \in \Re} |\Delta_T(x)| \geq \frac{1}{2} \eta - \frac{6}{T \pi \sqrt{2\pi}} h \left(\frac{T \sqrt{2\pi}}{2} \eta \right),$$

where $\eta \triangleq \sup_{x \in \Re} |H(x) - \Phi(x)|$, $\Delta_T(t) \triangleq \int_{-\infty}^{\infty} [H(t-x) - \Phi(t-x)] \times v_T(x) dx$,

and $h(u) \triangleq \pi u \int_u^{\infty} v_T \left(\frac{t}{T} \right) \frac{dt}{T} = \frac{\pi}{2} u + 1 - \cos(u) - u \int_0^u \frac{\sin(x)}{x} dx$.

Berry-Esseen theorem and probability bound

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Summary: η is the desired difference between any cdf $H(\cdot)$ and standard normal cdf $\Phi(\cdot)$.

$$\begin{aligned} & \sup_{x \in \mathbb{R}} \left| [H(x) - \Phi(x)] * v_T(x) \right| \\ & \geq \frac{1}{2}\eta - \frac{6}{T\pi\sqrt{2\pi}} \cdot \pi \cdot \frac{T\sqrt{2\pi}}{2}\eta \int_{T\sqrt{2\pi}\eta/2}^{\infty} v_T\left(\frac{t}{T}\right) \frac{dt}{T} \\ & = \frac{1}{2}\eta - 3\eta \int_{\sqrt{2\pi}\eta/2}^{\infty} v_T(y) dy \\ & = \eta \left(\frac{1}{2} - 3 \int_{\sqrt{2\pi}\eta/2}^{\infty} v_T(y) dy \right) \\ & = \left(\sup_{x \in \mathbb{R}} |H(x) - \Phi(x)| \right) \left(\frac{1}{2} - 3 \int_{\sqrt{2\pi}\eta/2}^{\infty} v_T(y) dy \right) \end{aligned}$$

Observation

- *The maximum absolute value of filter output bounds the maximum absolute value of filter input.*

Berry-Esseen theorem and probability bound

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Proof:

- The right-continuity of the cdf $H(\cdot)$ and the continuity of the Gaussian unit cdf $\Phi(\cdot)$ together indicate the right-continuity of $|H(x) - \Phi(x)|$, which in turn implies the existence of $x_0 \in \mathfrak{R}$ satisfying

$$\text{either } \eta = |H(x_0) - \Phi(x_0)| \quad \text{or} \quad \eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)|.$$

- We then distinguish between three cases:

$$\text{Case A) } \quad \eta = H(x_0) - \Phi(x_0);$$

$$\text{Case B) } \quad \eta = \Phi(x_0) - H(x_0);$$

$$\text{Case C) } \quad \eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)|.$$

Berry-Esseen theorem and probability bound

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Case A) $\eta = H(x_0) - \Phi(x_0)$.

In this case, we note that for $s > 0$,

$$H(x_0 + s) - \Phi(x_0 + s) \geq H(x_0) - \left[\Phi(x_0) + \frac{s}{\sqrt{2\pi}} \right] \quad (1)$$

$$= \eta - \frac{s}{\sqrt{2\pi}}, \quad (2)$$

where (1) follows from $\sup_{x \in \mathbb{R}} |\Phi'(x)| = 1/\sqrt{2\pi}$.

Observe that (2) implies

$$\begin{aligned} H\left(x_0 + \underbrace{\frac{\sqrt{2\pi}}{2} \eta - x}_{=s}\right) - \Phi\left(x_0 + \frac{\sqrt{2\pi}}{2} \eta - x\right) &\geq \eta - \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{2\pi}}{2} \eta - x \right) \\ &= \frac{1}{2} \eta + \frac{x}{\sqrt{2\pi}}, \end{aligned}$$

for $|x| < \eta\sqrt{2\pi}/2$.

Berry-Esseen theorem and probability bound

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Together with the fact that $H(x) - \Phi(x) \geq -\eta$ for all $x \in \mathfrak{R}$, we obtain

$$\begin{aligned}
 \sup_{x \in \mathfrak{R}} |\Delta_T(x)| &\geq \Delta_T \left(x_0 + \frac{\sqrt{2\pi}}{2} \eta \right) \\
 &= \int_{-\infty}^{\infty} \left[H \left(x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) - \Phi \left(x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) \right] \times v_T(x) dx \\
 &= \int_{|x| < \eta\sqrt{2\pi}/2} \left[H \left(x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) - \Phi \left(x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) \right] \times v_T(x) dx \\
 &+ \int_{|x| \geq \eta\sqrt{2\pi}/2} \left[H \left(x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) - \Phi \left(x_0 + \frac{\sqrt{2\pi}}{2} \eta - x \right) \right] \times v_T(x) dx \\
 &\geq \int_{|x| < \eta\sqrt{2\pi}/2} \left[\frac{1}{2}\eta + \frac{x}{\sqrt{2\pi}} \right] \times v_T(x) dx + \int_{|x| \geq \eta\sqrt{2\pi}/2} (-\eta) \times v_T(x) dx \\
 &= \int_{|x| < \eta\sqrt{2\pi}/2} \frac{1}{2}\eta \times v_T(x) dx + \int_{|x| \geq \eta\sqrt{2\pi}/2} (-\eta) \times v_T(x) dx,
 \end{aligned}$$

where the last equality holds because of the symmetry of the filtering function $v_T(\cdot)$.

Berry-Esseen theorem and probability bound

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The quantity of $\int_{|x| \geq \eta\sqrt{2\pi}/2} v_T(x) dx$ can be derived as follows:

$$\begin{aligned} \int_{|x| \geq \eta\sqrt{2\pi}/2} v_T(x) dx &= 2 \int_{\eta\sqrt{2\pi}/2}^{\infty} v_T(x) dx \\ &= 2 \int_{\eta T\sqrt{2\pi}/2}^{\infty} v_T\left(\frac{u}{T}\right) \frac{du}{T} \\ &= \frac{4}{\eta T \pi \sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta\right). \end{aligned}$$

Continuing from (3),

$$\begin{aligned} \sup_{x \in \mathfrak{R}} |\Delta_T(x)| &\geq \frac{1}{2} \eta \left[1 - \frac{4}{\eta T \pi \sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta\right) \right] \\ &\quad - \eta \cdot \left[\frac{4}{\eta T \pi \sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta\right) \right] \\ &= \frac{1}{2} \eta - \frac{6}{T \pi \sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta\right). \end{aligned}$$

Berry-Esseen theorem and probability bound

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Case B) $\eta = \Phi(x_0) - H(x_0)$.

Similar to Case A), we first derive for $s > 0$,

$$\Phi(x_0 - s) - H(x_0 - s) \geq \left[\Phi(x_0) - \frac{s}{\sqrt{2\pi}} \right] - H(x_0) = \eta - \frac{s}{\sqrt{2\pi}},$$

and then obtain

$$\begin{aligned} \Phi\left(x_0 - \frac{\sqrt{2\pi}}{2}\eta - x\right) - H\left(x_0 - \frac{\sqrt{2\pi}}{2}\eta - x\right) &\geq \eta - \frac{1}{\sqrt{2\pi}} \left(\underbrace{\frac{\sqrt{2\pi}}{2}\eta + x}_{=s} \right) \\ &= \frac{1}{2}\eta - \frac{x}{\sqrt{2\pi}}, \end{aligned}$$

for $|x| < \eta\sqrt{2\pi}/2$.

Berry-Esseen theorem and probability bound

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Together with the fact that $\Phi(x) - H(x) \geq -\eta$ for all $x \in \mathfrak{R}$, we obtain

$$\begin{aligned} \sup_{x \in \mathfrak{R}} |\Delta_T(x)| &\geq -\Delta_T \left(x_0 - \frac{\sqrt{2\pi}}{2} \eta \right) \\ &\geq \int_{[|x| < \eta\sqrt{2\pi}/2]} \left[\frac{1}{2}\eta - \frac{x}{\sqrt{2\pi}} \right] \times v_T(x) dx \\ &\quad + \int_{[|x| \geq \eta\sqrt{2\pi}/2]} (-\eta) \times v_T(x) dx \\ &= \int_{[|x| < \eta\sqrt{2\pi}/2]} \frac{1}{2}\eta \times v_T(x) dx + \int_{[|x| \geq \eta\sqrt{2\pi}/2]} (-\eta) \times v_T(x) dx \\ &= \frac{1}{2}\eta - \frac{6}{T\pi\sqrt{2\pi}} h \left(\frac{T\sqrt{2\pi}}{2} \eta \right). \end{aligned}$$

Berry-Esseen theorem and probability bound

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Case C) $\eta = \lim_{x \uparrow x_0} |H(x) - \Phi(x)| > |H(x_0) - \Phi(x_0)| \geq 0$.

In this case, we observe that for any $0 < \delta < \eta$, there exists x'_0 such that $|H(x'_0) - \Phi(x'_0)| \geq \eta - \delta \triangleq \eta'$. We can then follow the procedure of the previous two cases to obtain:

$$\sup_{x \in \mathfrak{R}} |\Delta_T(x)| \geq \frac{1}{2} \eta' - \frac{6}{T\pi\sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta'\right).$$

The proof is completed by noting that η' can be made arbitrarily close to η . \square

Berry-Esseen theorem and probability bound

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Lemma For any cumulative distribution function $H(\cdot)$ with zero-mean and unit variance, its characteristic function $\varphi_H(\zeta)$ satisfies that

$$\eta \leq \frac{1}{\pi} \int_{-T}^T \left| \varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right| \frac{d\zeta}{|\zeta|} + \frac{12}{T\pi\sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}\eta}{2}\right),$$

where η and $h(\cdot)$ are defined in the previous lemma.

This lemma transforms the bound in the previous lemma into *frequency domain*, namely the *characteristic function domain*.

Berry-Esseen theorem and probability bound

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Proof:

- Observe that

$$\Delta_T(t) = \int_{-\infty}^{\infty} [H(t-x) - \Phi(t-x)] \times v_T(x) dx$$

is nothing but a convolution of $v_T(\cdot)$ and $H(\cdot) - \Phi(\cdot)$.

- By Fourier inversion theorem,

$$\begin{aligned} \left. \frac{d(\Delta_T(t))}{dt} \right|_{t=x} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-j\zeta x} \left[\varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right] \omega_T(\zeta) d\zeta \\ &= \frac{1}{2\pi} \int_{-T}^T e^{-j\zeta x} \left[\varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right] \omega_T(\zeta) d\zeta, \end{aligned}$$

where the second step follows from bandlimit assumption.

Notably, $\varphi_H(\zeta)$ and $e^{-\zeta^2/2}$ are the Fourier transforms with respect to measures $dH(x)$ and $d\Phi(x)$, not the Fourier transforms of $H(x)$ and $\Phi(x)$.

Actually, $H(x)$ and $\Phi(x)$ have no Fourier transforms.

Berry-Esseen theorem and probability bound

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- Integrating with respect to x , we obtain

$$\Delta_T(x) = \frac{1}{2\pi} \int_{-T}^T e^{-j\zeta x} \frac{[\varphi_H(\zeta) - e^{-(1/2)\zeta^2}]}{-j\zeta} \omega_T(\zeta) d\zeta, \quad (3)$$

where no integration constant appears since both sides go to zero as $|x| \rightarrow \infty$.

Notably,

$$f'(x) = g'(x) \Rightarrow f(x) = g(x) + c,$$

where c is the integration constant.

Berry-Esseen theorem and probability bound

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Accordingly,

$$\begin{aligned} \sup_{x \in \mathfrak{R}} |\Delta_T(x)| &= \sup_{x \in \mathfrak{R}} \frac{1}{2\pi} \left| \int_{-T}^T e^{-j\zeta x} \frac{[\varphi_H(\zeta) - e^{-(1/2)\zeta^2}]}{-j\zeta} \omega_T(\zeta) d\zeta \right| \\ &\leq \sup_{x \in \mathfrak{R}} \frac{1}{2\pi} \int_{-T}^T \left| e^{-j\zeta x} \frac{[\varphi_H(\zeta) - e^{-(1/2)\zeta^2}]}{-j\zeta} \omega_T(\zeta) \right| d\zeta \\ &= \sup_{x \in \mathfrak{R}} \frac{1}{2\pi} \int_{-T}^T \left| \varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right| \cdot |\omega_T(\zeta)| \frac{d\zeta}{|\zeta|} \\ &\leq \sup_{x \in \mathfrak{R}} \frac{1}{2\pi} \int_{-T}^T \left| \varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right| \frac{d\zeta}{|\zeta|} \\ &= \frac{1}{2\pi} \int_{-T}^T \left| \varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right| \frac{d\zeta}{|\zeta|}. \end{aligned}$$

Berry-Esseen theorem and probability bound

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Together with

$$\sup_{x \in \mathfrak{R}} |\Delta_T(x)| \geq \frac{1}{2} \eta - \frac{6}{T\pi\sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta\right),$$

we finally have

$$\eta \leq \frac{1}{\pi} \int_{-T}^T \left| \varphi_H(\zeta) - e^{-(1/2)\zeta^2} \right| \frac{d\zeta}{|\zeta|} + \frac{12}{T\pi\sqrt{2\pi}} h\left(\frac{T\sqrt{2\pi}}{2} \eta\right).$$

□

Berry-Esseen theorem and probability bound

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Theorem Let $Y_n = \sum_{i=1}^n X_i$ be sum of i.i.d. random variables. Assume $n \geq 3$. Denote the mean and variance of X_1 by $\hat{\mu}$ and $\hat{\sigma}^2$, respectively.

Define

$$\hat{\rho} \triangleq E \left[|X_1 - \hat{\mu}|^3 \right].$$

Also denote the cdf of $(Y_n - E[Y_n]) / (\hat{\sigma}\sqrt{n})$ by $H_n(\cdot)$.

Then

$$\sup_{y \in \mathfrak{R}} |H_n(y) - \Phi(y)| \leq 5 \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}}.$$

Berry-Esseen theorem and probability bound

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Proof:

- $$\pi \cdot \eta \leq \int_{-T}^T \left| \hat{\varphi}^n \left(\frac{\zeta}{\hat{\sigma} \sqrt{n}} \right) - e^{-\zeta^2/2} \right| \frac{d\zeta}{|\zeta|} + \frac{12}{T\sqrt{2\pi}} h \left(\frac{T\sqrt{2\pi}}{2} \eta \right), \quad (4)$$

where $\hat{\varphi}(\cdot)$ is the characteristic function of $(X_1 - \hat{\mu})$.

Lemma For any complex numbers α and β ,

$$|\alpha^n - \beta^n| \leq n|\alpha - \beta|\gamma^{n-1},$$

where $\gamma \geq \max\{|\alpha|, |\beta|\}$.

Proof:

$$\begin{aligned} |\alpha^n - \beta^n| &= \left| \left(\frac{\alpha}{r} \right)^n - \left(\frac{\beta}{r} \right)^n \right| r^n \\ &\leq n \left| \frac{\alpha}{r} - \frac{\beta}{r} \right| r^n \\ &= n|\alpha - \beta| r^{n-1}. \end{aligned}$$

■

Berry-Esseen theorem and probability bound

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- Observe that the integrand satisfies

$$\begin{aligned} & \left| \hat{\varphi}^n \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - e^{-\zeta^2/2} \right| \\ & \leq n \left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - e^{-\zeta^2/(2n)} \right| \gamma^{n-1}, \end{aligned} \quad (5)$$

$$\leq n \left(\left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - \left(1 - \frac{\zeta^2}{2n} \right) \right| + \left| \left(1 - \frac{\zeta^2}{2n} \right) - e^{-\zeta^2/(2n)} \right| \right) \gamma^{n-1} \quad (6)$$

where the quantity γ in (5) requires that

$$\left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) \right| \leq \gamma \quad \text{and} \quad \left| e^{-\zeta^2/(2n)} \right| \leq \gamma.$$

We upperbound the first and second terms in the parentheses of (6) respectively by

$$\left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - 1 + \frac{\zeta^2}{2n} \right| \leq \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta|^3 \quad \text{and} \quad \left| 1 - \frac{\zeta^2}{2n} - e^{-\zeta^2/(2n)} \right| \leq \frac{1}{8n^2} \zeta^4.$$

Continuing the derivation of (6),

$$\left| \hat{\varphi}^n \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - e^{-\zeta^2/2} \right| \leq n \left(\frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta|^3 + \frac{1}{8n^2} \zeta^4 \right) \gamma^{n-1}. \quad (7)$$

It remains to choose γ that bounds both $|\hat{\varphi}(\zeta/(\hat{\sigma}\sqrt{n}))|$ and $\exp\{-\zeta^2/(2n)\}$ from above.

Berry-Esseen theorem and probability bound

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For complex number z and reals b and c ,

$$|z - b| \leq c \Rightarrow |z| \leq |b| + c.$$

Accordingly,

$$\left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - 1 + \frac{\zeta^2}{2n} \right| \leq \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta|^3 \Rightarrow \left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) \right| \leq \left| 1 - \frac{\zeta^2}{2n} \right| + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta|^3.$$

- Next,

$$\left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) \right| \leq 1 - \frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta|^3, \text{ if } \frac{\zeta^2}{2n} \leq 1. \quad (8)$$

For those $\zeta \in [-T, T]$ (which is exactly the range of integration operation in (4)), we can guarantee the validity of the condition in (8) by defining

$$T \triangleq \frac{\hat{\sigma}^3 \sqrt{n}}{\hat{\rho}} \left(\frac{\sqrt{2n} - 3}{n - 1} \right),$$

and obtain

$$\frac{\zeta^2}{2n} \leq \frac{T^2}{2n} = \frac{\hat{\sigma}^6}{2\hat{\rho}^2} \left(\frac{\sqrt{2n} - 3}{n - 1} \right)^2 \leq \frac{1}{2} \left(\frac{\sqrt{2n} - 3}{n - 1} \right)^2 \leq 1,$$

for $n \geq 3$.

Berry-Esseen theorem and probability bound

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Hence, for $|\zeta| \leq T$,

$$\begin{aligned} \left| \hat{\varphi} \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) \right| &\leq 1 + \left(-\frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta^3| \right) \\ &\leq \exp \left\{ -\frac{\zeta^2}{2n} + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} |\zeta^3| \right\} \\ &\leq \exp \left\{ -\frac{1}{2n} \zeta^2 + \frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} T \zeta^2 \right\} \\ &= \exp \left\{ -\left(\frac{1}{2n} - \frac{\hat{\rho}T}{6\hat{\sigma}^3 n^{3/2}} \right) \zeta^2 \right\} \\ &= \exp \left\{ -\frac{(3 - \sqrt{2})}{6(n-1)} \zeta^2 \right\}. \end{aligned}$$

We can then choose

$$\gamma \triangleq \exp \left\{ -\frac{(3 - \sqrt{2})}{6(n-1)} \zeta^2 \right\}.$$

Note that the above selected γ is an upper bound of $\exp \{-\zeta^2/(2n)\}$ for $n \geq 3/\sqrt{2} \approx 2.12$.

Berry-Esseen theorem and probability bound

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- By taking the chosen γ into (7), the integration part in (4) becomes

$$\begin{aligned}
 & \int_{-T}^T \left| \hat{\varphi}^n \left(\frac{\zeta}{\hat{\sigma}\sqrt{n}} \right) - e^{-\zeta^2/2} \right| \frac{d\zeta}{|\zeta|} \\
 & \leq \int_{-T}^T n \left(\frac{\hat{\rho}}{6\hat{\sigma}^3 n^{3/2}} \zeta^2 + \frac{1}{8n^2} |\zeta|^3 \right) \cdot \exp \left\{ -\frac{(3-\sqrt{2})}{6} \zeta^2 \right\} d\zeta \\
 & \leq \int_{-\infty}^{\infty} \left(\frac{\hat{\rho}}{6\hat{\sigma}^3 \sqrt{n}} \zeta^2 + \frac{1}{8n} |\zeta|^3 \right) \cdot \exp \left\{ -\frac{(3-\sqrt{2})}{6} \zeta^2 \right\} d\zeta \\
 & = \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}} \left(\frac{\sqrt{6\pi}}{2(3-\sqrt{2})^{3/2}} + \frac{9}{2(3-\sqrt{2})^2} \frac{\hat{\sigma}^3}{\hat{\rho}\sqrt{n}} \right) \\
 & \leq \frac{\hat{\rho}}{\hat{\sigma}^3 \sqrt{n}} \left(\frac{\sqrt{6\pi}}{2(3-\sqrt{2})^{3/2}} + \frac{9}{2(3-\sqrt{2})^2} \frac{1}{\sqrt{n}} \right) \\
 & = \frac{1}{T} \left(\frac{\sqrt{2}n-3}{n-1} \right) \left(\frac{\sqrt{6\pi}}{2(3-\sqrt{2})^{3/2}} + \frac{9}{2(3-\sqrt{2})^2} \frac{1}{\sqrt{n}} \right), \tag{9}
 \end{aligned}$$

where the last inequality follows from Lyapounov's inequality, i.e.,

$$\hat{\sigma} = E^{1/2} \left[|X_{d+1} - \hat{\mu}|^2 \right] \leq E^{1/3} \left[|X_{d+1} - \hat{\mu}|^3 \right] = \hat{\rho}^{1/3}.$$

Berry-Esseen theorem and probability bound

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- Taking (9) into (4), we finally obtain

$$\begin{aligned} \pi \cdot \eta &\leq \frac{1}{T} \left(\frac{\sqrt{2}n - 3}{n - 1} \right) \left(\frac{\sqrt{6\pi}}{2(3 - \sqrt{2})^{3/2}} + \frac{9}{2(3 - \sqrt{2})^2} \frac{1}{\sqrt{n}} \right) \\ &\quad + \frac{12}{T\sqrt{2\pi}} h \left(\frac{T\sqrt{2\pi}}{2} \eta \right), \end{aligned}$$

or equivalently,

$$\pi u - 6h(u) \leq \frac{\sqrt{\pi}(2n - 3\sqrt{2})}{4(n - 1)} \left(\frac{\sqrt{6\pi}}{(3 - \sqrt{2})^{3/2}} + \frac{9}{(3 - \sqrt{2})^2} \frac{1}{\sqrt{n}} \right), \quad (10)$$

for $u \triangleq T\sqrt{2\pi}\eta/2$.

Berry-Esseen theorem and probability bound

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Observe that

$$\pi u - 6h(u) = \pi u \left(1 - 6 \int_u^\infty v_T \left(\frac{t}{T} \right) \frac{dt}{T} \right)$$

is continuous, and equals 0 at $u = 0$, and goes to ∞ as $u \rightarrow \infty$, which guarantees the existence of positive u satisfying (10).

Inequality (10) thus implies

$$u \leq \max \left\{ a \geq 0 : \pi a - 6h(a) \leq \frac{\sqrt{\pi}(2n - 3\sqrt{2})}{4(n-1)} \left(\frac{\sqrt{6\pi}}{(3 - \sqrt{2})^{3/2}} + \frac{9}{(3 - \sqrt{2})^2} \frac{1}{\sqrt{n}} \right) \right\}.$$

Since

$$\frac{\sqrt{\pi}(2n - 3\sqrt{2})}{4(n-1)} = \frac{\sqrt{\pi}}{4} \left(2 - \frac{3\sqrt{2} - 2}{n-1} \right) \leq \frac{\sqrt{\pi}}{2},$$

we can, for $n \geq 3$, further upper-bound u by:

$$\begin{aligned} u &\leq \max \left\{ a \geq 0 : \pi a - 6h(a) \leq \frac{\sqrt{\pi}}{2} \left(\frac{\sqrt{6\pi}}{(3 - \sqrt{2})^{3/2}} + \frac{9}{(3 - \sqrt{2})^2} \frac{1}{\sqrt{3}} \right) \right\} \\ &= 3.26369\dots \end{aligned}$$

Berry-Esseen theorem and probability bound

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The proof is completed by

$$\begin{aligned}\eta &= u \frac{2}{T\sqrt{2\pi}} \\ &\leq 3.26369 \frac{2}{T\sqrt{2\pi}} \\ &= 6.52738 \frac{(n-1)}{\sqrt{\pi}(2n-3\sqrt{2})} \frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\ &= \frac{6.52738}{2\sqrt{\pi}} \left(1 + \frac{3\sqrt{2}-2}{2n-3\sqrt{2}}\right) \frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\ &\leq \frac{6.52738}{2\sqrt{\pi}} \left(1 + \frac{3\sqrt{2}-2}{6-3\sqrt{2}}\right) \frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\ &= 4.19115 \frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}} \\ &\leq 5 \frac{\hat{\rho}}{\hat{\sigma}^3\sqrt{n}}.\end{aligned}$$

□