

# Ninness's Strong Law of Large Numbers

Po-Ning Chen, Professor

Institute of Communications Engineering

National Chiao Tung University

Hsin Chu, Taiwan 30010, R.O.C.

## The law of large numbers

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- The employment of “Strong Law of Large Numbers (SLLN)” is instrumental to the analysis of system estimation and identification strategies.
- However, the condition (for its validity) such as independence or uncorrelatedness of random components is quite restrictive from an engineering standpoint.
- In his paper, Brett Ninness shows that the SLLN is valid even for possibly non-stationary random components under very general dependence structure.
  - B. Ninness, “Strong Laws of Large Numbers Under Weak Assumptions with Applications,” *IEEE Transactions on Automatic Control*, Vol. 45, No. 11, pp. 2117-2122, 2000.

# Ninness Strong Law of Large Numbers

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**Theorem** Suppose  $\{X_n\}_{n=1}^{\infty}$  is a sequence of random variables, not necessarily zero mean, and with arbitrary correlation structure (not necessarily stationary) that is characterized by the existence of  $C < \infty$  and  $\beta > 1$  such that

$$\sum_{k=i+1}^j \sum_{\ell=i+1}^j E[X_k X_\ell] \leq C(j-i)^\beta \text{ for every } 0 \leq i \leq j.$$

Then for any  $\alpha > \beta/2$ ,

$$\frac{1}{n^\alpha} \sum_{k=1}^n X_k \xrightarrow{a.s.} 0 \text{ as } n \rightarrow \infty.$$

**Proof:**

- Define  $S_n = \sum_{k=1}^n X_k$ , and for any  $n$ , choose an integer  $m = \lfloor \log_2(n) \rfloor$  such that

$$2^m \leq n < 2^{m+1}.$$

Then

$$\frac{1}{n^\alpha} |S_n| \leq \frac{1}{2^{m\alpha}} \max_{0 \leq k \leq 2^{m+1}} |S_k|.$$

## Ninness Strong Law of Large Numbers

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Refined version of **Theorem 12.2** in [P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, New York, 1968]. For arbitrary random variables  $\{X_n\}_{n=1}^{\infty}$ , if for some  $\beta > 1$ , there exists a set of non-negative numbers  $\{u_n\}_{n=1}^{\infty}$  such that

$$E \left[ |S_j - S_i|^2 \right] \leq \left( \sum_{k=i}^j u_k \right)^{\beta} \quad \text{for every } 0 \leq i \leq j \leq N,$$

where  $S_j = \sum_{k=1}^j X_k$  and  $S_0 = 0$ , then there exists a  $K < \infty$  that is independent of  $N$  such that for any  $\lambda > 0$ ,

$$\Pr \left\{ \max_{0 \leq \ell \leq N} |S_{\ell}| \geq \lambda \right\} \leq \frac{K}{\lambda^2} \left( \sum_{k=1}^N u_k \right)^{\beta}.$$

Furthermore, the constant  $K$  can be taken as  $K = 4 \left( 1 + \frac{2^{(\beta-1)}}{(2^{(\beta-1)/3} - 1)^3} \right)$ .

- Taking  $u_n = C^{1/\beta}$  for every  $n$  in the above refined theorem yields:

# Ninness Strong Law of Large Numbers

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If for some  $\beta > 1$ ,

$$E \left[ |S_j - S_i|^2 \right] \leq C (j - i)^\beta \text{ for every } 0 \leq i \leq j,$$

then there exists a  $K < \infty$  that is only dependent on  $\beta$  such that for any  $\lambda > 0$  and any  $N$ ,

$$\Pr \left\{ \max_{0 \leq \ell \leq N} |S_\ell| \geq \lambda \right\} \leq \frac{KC}{\lambda^2} N^\beta.$$

Hence for  $\lambda = \varepsilon 2^{m\alpha}$  and  $N = 2^{m+1}$ ,

$$\Pr \left\{ \max_{0 \leq k \leq 2^{m+1}} |S_k| \geq \varepsilon 2^{m\alpha} \right\} \leq \frac{KC}{\varepsilon^2 2^{2m\alpha}} (2^{m+1})^\beta = \frac{KC 2^\beta}{\varepsilon^2 2^{(2\alpha - \beta)m}}$$

since

$$E[|S_j - S_i|^2] = \sum_{k=i+1}^j \sum_{\ell=i+1}^j E[X_k X_\ell] \leq C (j - i)^\beta$$

for every  $0 \leq i \leq j$ .

## Ninness Strong Law of Large Numbers

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- As a result, for  $\alpha > \beta/2$ ,

$$\sum_{m=1}^{\infty} \Pr \left\{ \max_{0 \leq k \leq 2^{m+1}} |S_k| \geq \varepsilon 2^{m\alpha} \right\} \leq \frac{KC2^\beta}{\varepsilon^2} \sum_{m=1}^{\infty} 2^{-(2\alpha-\beta)m} = \frac{KC2^\beta}{\varepsilon^2(2^{2\alpha-\beta} - 1)} < \infty.$$

By the first Borel-Cantelli lemma, we obtain that with probability 1

$$\frac{1}{2^{m\alpha}} \max_{0 \leq k \leq 2^{m+1}} |S_k| \geq \varepsilon$$

is valid only for finitely many  $m$ .

**Theorem 4.3 (The First Borel-Cantelli Lemma)** If  $\sum_{n=1}^{\infty} P(A_n)$  converges, then

$$P \left( \limsup_{n \rightarrow \infty} A_n \right) = P(A_n \text{ i.o.}) = 0.$$

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- Since  $\varepsilon$  can be made arbitrarily small,

$$\limsup_{m \rightarrow \infty} \frac{1}{2^{m\alpha}} \max_{0 \leq k \leq 2^{m+1}} |S_k| = 0 \text{ with probability 1.}$$

The theorem holds by noting that

$$\limsup_{n \rightarrow \infty} \frac{1}{n^\alpha} |S_n| \leq \limsup_{n \rightarrow \infty} \frac{1}{2^{\lfloor \log_2(n) \rfloor \alpha}} \max_{0 \leq k \leq 2^{\lfloor \log_2(n) \rfloor + 1}} |S_k| = \limsup_{m \rightarrow \infty} \frac{1}{2^{m\alpha}} \max_{0 \leq k \leq 2^{m+1}} |S_k|.$$

□

## Example 1

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- Let

$$y_n = \theta + v_n,$$

where  $\theta$  is an unknown constant to be estimated based upon the observations of  $\{y_n\}$ , and

$$E[v_n v_m] = \frac{1 + |n - m|^p}{1 + |n - m|^q} \text{ for some } q \geq 1 \text{ and } q - 1 \leq p < q.$$

- Use the estimator  $\hat{\theta} = \frac{1}{n} \sum_{k=1}^n y_k$ .
- Then the estimation error  $\theta_e = \hat{\theta} - \theta = \frac{1}{n} \sum_{k=1}^n v_k$ .
- Verify by defining  $S_j = v_1 + v_2 + \cdots + v_j$  that

$$\begin{aligned} E[|S_j - S_i|^2] &= \sum_{k=i+1}^j \sum_{\ell=i+1}^j E[v_k v_\ell] \\ &= \sum_{k=i+1}^j \sum_{\ell=i+1}^j \frac{1 + |k - \ell|^p}{1 + |k - \ell|^q} \\ &= \sum_{k=1}^m \sum_{\ell=1}^m \frac{1 + |k - \ell|^p}{1 + |k - \ell|^q} \quad (m = j - i) \end{aligned}$$



## Example 1

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$$\begin{aligned} &= m + 2 \sum_{k=1}^m \sum_{\ell=1}^{k-1} \frac{1 + (k - \ell)^p}{1 + (k - \ell)^q} \\ &\leq m + 2 \sum_{k=1}^m \sum_{\ell=1}^{k-1} \frac{2(k - \ell)^p}{1 + (k - \ell)^q} \quad (\text{since } p \geq 0) \\ &\leq m + 4 \sum_{k=1}^m \sum_{\ell=1}^{k-1} \frac{m^p}{1 + (k - \ell)^q} \quad (\text{since } \max_{1 \leq k \leq m} \max_{1 \leq \ell \leq k-1} (k - \ell) = (m - 1) \leq m) \\ &= m + 4m^p \sum_{k=1}^m \sum_{\ell=1}^{k-1} \frac{1}{1 + (k - \ell)^q} \\ &= m + 4m^p \sum_{u=1}^m (m - u) \cdot \frac{1}{1 + u^q} \\ &\leq m + 4m^{p+1} \sum_{u=1}^m \frac{1}{1 + u^q} \quad (\text{because } m - u \leq m \text{ for } 1 \leq u \leq m) \\ &\leq m + 4m^{p+1} \int_0^m \frac{1}{1 + u^q} du \quad (\text{because } 1 + u^q \text{ nonincreasing for } q \geq 0) \end{aligned}$$

## Example 1

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$$\begin{aligned} &\leq m + 4m^{p+1} \int_0^m \frac{1}{1+u^q} du \\ &= m + 4m^{p+1} \left( \int_0^1 \frac{1}{1+u^q} du + \int_1^m \frac{1}{1+u^q} du \right) \\ &\leq m + 4m^{p+1} \left( 1 + \int_1^m \frac{1}{1+u^q} du \right) \quad (\text{since } \int_0^1 \frac{1}{1+u^q} du \leq 1) \\ &\leq m + 4m^{p+1} \left( 1 + \int_1^m \frac{1}{u^q} du \right) \\ &= m + 4m^{p+1} \left( \frac{m^{1-q} - q}{1-q} \right) \\ &= m + \frac{4}{1-q} m^{p-q+2} - \frac{4q}{1-q} m^{p+1} \\ &\leq m^{p-q+2} + \frac{4}{1-q} m^{p-q+2} - \frac{4q}{1-q} m^{p-q+2} \quad (\text{because } q \geq 1 \text{ and } p \geq q-1) \\ &= \frac{5}{1-q} m^{p-q+2}. \end{aligned}$$

## Example 1

N-10

Hence, as  $(\alpha =)1 > (p - q + 2)/2(= \beta/2)$ ,

$$\theta_e = \hat{\theta} - \theta = \frac{1}{n^\alpha} \sum_{k=1}^n v_n = \frac{1}{n} \sum_{k=1}^n v_n \rightarrow 0 \text{ with probability 1.}$$

## Example 2

N-11

- Suppose  $\{v_n\}$  (in Example 1) is wide-sense stationary with bounded power spectral density. Then, by denoting the bound for the power spectral density by  $P$ ,

$$\begin{aligned} E[|S_j - S_i|^2] &= \sum_{k=i+1}^j \sum_{\ell=i+1}^j E[v_k v_\ell] \\ &= \sum_{k=1}^m \sum_{\ell=1}^m E[v_k v_\ell] \quad (m = j - i) = \sum_{k=1}^m \sum_{\ell=1}^m R_v(k - \ell) \\ &\leq \sum_{k=1}^m \sum_{\ell=1}^m \int_{-1/2}^{1/2} P e^{i2\pi(k-\ell)f} df \\ &= P \sum_{k=1}^m \sum_{\ell=1}^m \int_{-1/2}^{1/2} e^{i2\pi(k-\ell)f} df \\ &= Pm. \end{aligned}$$

- Hence, for  $\alpha = 1 > 1/2 = \beta/2$ ,

$$\theta_e = \hat{\theta} - \theta = \frac{1}{n} \sum_{k=1}^n v_n \rightarrow 0 \text{ with probability 1.}$$

## Example 2

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$$\left\{ \begin{array}{l} S_v(\omega) = \sum_{k=-\infty}^{\infty} R_v[k] e^{-i\omega k} \\ R_v[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_v(\omega) e^{i\omega k} d\omega \end{array} \right. \equiv \left\{ \begin{array}{l} S_v(f) = \sum_{k=-\infty}^{\infty} R_v[k] e^{-i2\pi k f} \\ R_v[k] = \int_{-1/2}^{1/2} S_v(f) e^{i2\pi k f} df \end{array} \right.$$