Section 22

Sums of Independent Random Variables

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Law of large numbers revisited

Theorem 22.1 (advanced version of strong law of large numbers) If $X_1, X_2, \ldots$ are **pair-wise** independent with common marginal distribution and finite mean, then

$$\frac{S_n}{n} \to E[X_1] \quad \text{with probability 1,}$$

where $S_n = X_1 + X_2 + \cdots + X_n$.

Proof (due to Etemadi): Assume without loss of generality that X_i is non-negative.

If the theorem holds for non-negative random variables, then

$$\frac{S_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k^+ - \frac{1}{n}\sum_{k=1}^{n} X_k^- \stackrel{w.p.1}{\longrightarrow} E[X_1^+] - E[X_1^-] = E[X_1].$$

- Consider the truncated random variable $Y_k = X_k I_{[X_k \le k]}$, and denote $S_n^* = \sum_{k=1}^{n} Y_k$. (Notably, $Y_1, Y_2, \ldots$ is not identically distributed, but only pair-wise independent.)

  Then for $k \le n$,

$$E[Y_k^2] = E[X_k^2 I_{[X_k \le k]}] = E[X_1^2 I_{[X_1 \le k]}] \le E[X_1^2 I_{[X_1 \le n]}] = E[Y_n^2].$$

Law of large numbers revisited

The reason of introducing a truncated version of X_n is because $E[X_n^2]$ may be infinity! This is the key technique used in this proof.

- *Claim:* For $u_n \triangleq \lfloor \alpha^n \rfloor$ with $\alpha > 1$ fixed,

$$\sum_{n=1}^{\infty} \Pr\left[\left|\frac{S_{u_n}^* - E[S_{u_n}^*]}{u_n}\right| > \varepsilon\right] < \infty \quad \text{for any } \varepsilon > 0.$$

Theorem 4.3 (First Borel-Cantelli lemma)

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = P(A_n \text{ i.o.}) = 0.$$

Proof of the claim: By Chebyshev's inequality,

$$\sum_{n=1}^{\infty} \Pr\left[\left|\frac{S_{u_n}^* - E[S_{u_n}^*]}{u_n}\right| > \varepsilon\right] \le \sum_{n=1}^{\infty} \frac{\mathrm{Var}[S_{u_n}^*]}{u_n^2 \varepsilon^2} \le \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{E[Y_{u_n}^2]}{u_n},$$

where by pair-wise independence,

$$\mathrm{Var}[S_{u_n}^*] = \sum_{k=1}^{u_n} \mathrm{Var}[Y_k] \le u_n E[Y_{u_n}^2].$$

Law of large numbers revisited

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}\Pr\left[\left|\frac{S^*_{u_n}-E[S^*_{u_n}]}{u_n}\right|>\varepsilon\right] &\leq \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{E[Y^2_{u_n}]}{u_n}=\frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{E[X^2_{u_n}I_{[X_{u_n}\leq u_n]}]}{u_n}\\
&= \frac{1}{\varepsilon^2}\sum_{n=1}^{\infty}\frac{E[X_1^2 I_{[X_1\leq u_n]}]}{u_n}=\frac{1}{\varepsilon^2}\lim_{m\to\infty}\sum_{n=1}^{m}\frac{E[X_1^2 I_{[X_1\leq u_n]}]}{u_n}\\
&= \frac{1}{\varepsilon^2}\lim_{m\to\infty}E\left[X_1^2\sum_{n=1}^{m}\frac{1}{u_n}I_{[X_1\leq u_n]}\right] \quad \left(f_m(x)\triangleq x^2\sum_{n=1}^{m}\frac{1}{u_n}I_{[x\leq u_n]}\right)\\
&= \frac{1}{\varepsilon^2}E\left[X_1^2\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{u_n}I_{[X_1\leq u_n]}\right] \quad \text{(by monotone conv. thm.)}\\
&= \frac{1}{\varepsilon^2}E\left[X_1^2\sum_{n=1}^{\infty}\frac{1}{u_n}I_{[X_1\leq u_n]}\right]\end{aligned}$$

Monotone convergence theorem: If for every positive integer m and every x in the support $\mathcal{X}$ of random variable X, $0\leq f_m(x)\leq f_{m+1}(x)$, then

$$\lim_{m\to\infty}E[f_m(X)]=\lim_{m\to\infty}\int_{\mathcal{X}}f_m(x)dP_X(x)=\int_{\mathcal{X}}\lim_{m\to\infty}f_m(x)dP_X(x)=E\left[\lim_{m\to\infty}f_m(X)\right].$$

Law of large numbers revisited

Observe that for any $x > 0$ fixed,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{u_n} I_{[x \le u_n]} &= \sum_{\{n \in \mathbb{N} \,:\, u_n \ge x\}} \frac{1}{u_n} \\
&= \sum_{n \ge N} \frac{1}{u_n}, \text{ where } N = \min\{n \in \mathbb{N} : u_n \ge x\} \\
&\le \sum_{n \ge N} \frac{2}{\alpha^n}, \text{ (since } u_n = \lfloor \alpha^n \rfloor \text{ and } \lfloor y \rfloor \ge \frac{1}{2} y \text{ for } y \ge 1) \\
&= \left(\frac{2}{1 - \alpha^{-1}} \right) \frac{1}{\alpha^N} \\
&\le \left(\frac{2\alpha}{\alpha - 1} \right) \frac{1}{x}. \quad \text{(by } \alpha^N \ge \lfloor \alpha^N \rfloor = u_N \ge x)
\end{aligned}$$

This concludes that:

$$\sum_{n=1}^{\infty} \Pr\left[\left| \frac{S^*_{u_n} - E[S^*_{u_n}]}{u_n} \right| > \varepsilon \right] \le \frac{1}{\varepsilon^2} E\left[X_1^2 \sum_{n=1}^{\infty} \frac{1}{u_n} I_{[X_1 \le u_n]} \right] \le \frac{1}{\varepsilon^2} \left(\frac{2\alpha}{\alpha - 1} \right) E\left[X_1 \right] < \infty.$$

Law of large numbers revisited

- By the above claim and the first Borel-Cantelli lemma,

$$\frac{S^*_{u_n} - E[S^*_{u_n}]}{u_n} \to 0 \text{ with probability 1.}$$

- By the Cesáro-mean theorem (cf. the next slide),

$$\lim_{u_n\to\infty} E[Y_{u_n}] \quad \left(= \lim_{n\to\infty} E[Y_{u_n}] = \lim_{n\to\infty} E[X_1 I_{[X_1 \le u_n]}] \right) \quad = E[X_1] < \infty$$

implies

$$\frac{1}{u_n} E[S^*_{u_n}] = \frac{1}{u_n} \sum_{k=1}^{u_n} E[Y_k] \to E[X_1] \quad \text{as } n \to \infty.$$

Thus,

$$\frac{S^*_{u_n}}{u_n} \to E[X_1] \text{ with probability 1.}$$

Law of large numbers revisited

Theorem (Cesáro-mean theorem) If $\lim_{n\to\infty} a_n = a$ and $b_n = (1/n)\sum_{i=1}^{n} a_i$, where a is finite, then $\lim_{n\to\infty} b_n = a$.

Proof: $\lim_{n\to\infty} a_n = a$ implies that for any $\varepsilon > 0$, there exists N such that for all $n > N$, $|a_n - a| < \varepsilon$. Then

$$\begin{aligned}
|b_n - a| &= \left|\frac{1}{n}\sum_{i=1}^{n}(a_i - a)\right| \le \frac{1}{n}\sum_{i=1}^{n}|a_i - a| \\
&= \frac{1}{n}\sum_{i=1}^{N}|a_i - a| + \frac{1}{n}\sum_{i=N+1}^{n}|a_i - a| \\
&\le \frac{1}{n}\sum_{i=1}^{N}|a_i - a| + \frac{n-N}{n}\varepsilon.
\end{aligned}$$

Hence, $\lim_{n\to\infty}|b_n - a| \le \varepsilon$. Since ε can be made arbitrarily small, $\lim_{n\to\infty} b_n = a$. □

Law of large numbers revisited

- *Claim*: $\dfrac{S_n - S_n^*}{n} \to 0$ with probability 1.

  Proof of the claim:

$$
\begin{aligned}
\sum_{n=1}^{\infty} \Pr[X_n \neq Y_n] &= \sum_{n=1}^{\infty} \Pr[X_n \neq X_n I_{[X_n \leq n]}] \\
&= \sum_{n=1}^{\infty} \Pr[X_n > n] \\
&= \sum_{n=1}^{\infty} \Pr[X_1 > n] \text{ (by ``identical distributed'' assumption)} \\
&\leq \int_0^{\infty} \Pr[X_1 > t] dt \\
&= E[X_1] \text{ (by non-negativity assumption of } X_1\text{)} \\
&< \infty.
\end{aligned}
$$

  Hence, the first Bore-Cantelli lemma gives that

$$
\Pr[(X_n \neq Y_n) \text{ is true infinitely often in } n] = 0,
$$

  equivalently,

$$
\Pr[(X_n \neq Y_n) \text{ is true finitely many in } n] = 1.
$$

Law of large numbers revisited

This implies that

$$\Pr\left[(\exists\ \mathbb{U} = \{n_1, n_2, \ldots, n_M\})\ X_n \neq Y_n \text{ only for } n \in \mathbb{U}\right] = 1.$$

The above result, together with the fact that

$$\Pr[(X_n - Y_n) < \infty] = \Pr\left[X_n I_{[X_n > n]} < \infty\right] = \Pr\left[X_1 I_{[X_1 > n]} < \infty\right] = 1$$

because $E[X_1] < \infty$, leads to:

$$\begin{aligned} & \Pr\left[\lim_{n\to\infty} \frac{(X_1 - Y_1) + \cdots + (X_n - Y_n)}{n} = 0\right] \\ = & \Pr\left[\lim_{n\to\infty} \frac{(X_{n_1} - Y_{n_1}) + \cdots + (X_{n_M} - Y_{n_M})}{n} = 0\right] \\ = & 1. \end{aligned}$$

- Now we have
  $S^*_{u_n}/u_n \to E[X_1]$ with probability 1, where $u_n = \lfloor \alpha^n \rfloor$ for some $\alpha > 1$ fixed, and
  $(S_n - S^*_n)/n \to 0$ with probability 1.

The above two results directly imply $S_{u_n}/u_n \to E[X_1]$ (as n goes to infinity) with probability 1.

It remains to show $S_k/k \to E[X_1]$ (as k goes to infinity) with probability 1.

Law of large numbers revisited

- For $u_n \le k < u_{n+1}$,

$$
\begin{aligned}
\boxed{\frac{u_n}{u_{n+1}}\frac{S_{u_n}}{u_n}} &= \frac{S_{u_n}}{u_{n+1}} \\
&= \frac{X_1 + \cdots + X_{u_n}}{u_{n+1}} \\
&\le \frac{X_1 + \cdots + X_{u_n}}{k} \\
&\le \frac{X_1 + \cdots + X_{u_n} + \cdots + X_k}{k} = \boxed{\frac{S_k}{k}} \\
&\le \frac{X_1 + \cdots + X_{u_n} + \cdots + X_k}{u_n} \\
&\le \frac{X_1 + \cdots + X_{u_n} + \cdots + X_k + \cdots + X_{u_{n+1}}}{u_n} \\
&= \frac{S_{u_{n+1}}}{u_n} \\
&= \boxed{\frac{u_{n+1}}{u_n}\frac{S_{u_{n+1}}}{u_{n+1}}},
\end{aligned}
$$

since X_n is assumed non-negative.

Law of large numbers revisited

Because

$$\frac{u_n}{u_{n+1}} \frac{S_{u_n}}{u_n} \to \frac{1}{\alpha} E[X_1] \text{ with probability 1,}$$

and

$$\frac{u_{n+1}}{u_n} \frac{S_{u_{n+1}}}{u_{n+1}} \to \alpha E[X_1] \text{ with probability 1,}$$

we obtain:

$$\frac{1}{\alpha} E[X_1] \leq \liminf_{k\to\infty} \frac{S_k}{k} \leq \limsup_{k\to\infty} \frac{S_k}{k} \leq \alpha E[X_1] \text{ with probability 1.}$$

As the above statement is valid for any $\alpha > 1$, we conclude that

$$\frac{S_k}{k} \to E[X_1] \text{ with probability 1.}$$

□

Law of large numbers revisited

Theorem If $X_1, X_2, \ldots$ are **pair-wise** independent with common marginal distribution whose mean exists (could be infinity as defined in Slide 21-1), then

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to E[X_1] \text{ with probability 1.}$$

Proof: Now, based on the previous theorem, we only need to prove the current theorem for the case of $E[X_1] = \infty$.

- Suppose without loss of generality that $E[X_1^-] < \infty$ and $E[X_1^+] = \infty$. Then

$$\frac{1}{n}\sum_{k=1}^{n} X_k^- \to E[X_1^-] \text{ with probability 1.}$$

- Let $Y_n(u) = X_n^+ I_{[X_n \le u]}$, and observe that

$$\frac{1}{n}\sum_{k=1}^{n} X_k^+ \ge \frac{1}{n}\sum_{k=1}^{n} Y_k(u), \text{ and } \frac{1}{n}\sum_{k=1}^{n} Y_k(u) \to E[Y_k(u)] \text{ with probability 1.}$$

  Hence,

$$\frac{1}{n}\sum_{k=1}^{n} X_k^+ \ge E[Y_k(u)] \text{ (as } n \text{ goes to infinity) with probability 1.}$$

Law of large numbers revisited

- Since the above statement is valid for any u, and $E[Y_k(u)] \to \infty$ as $u \to \infty$,

$$\frac{1}{n}\sum_{k=1}^{n} X_k^+ \to \infty \text{ with probability 1.}$$

- Finally,

$$\frac{1}{n}\sum_{k=1}^{n} X_k = \frac{1}{n}\sum_{k=1}^{n} X_k^+ - \frac{1}{n}\sum_{k=1}^{n} X_k^- \to \infty \text{ with probability 1.}$$

□

Limit of normalized Poisson

Next, we introduce a famous result for Possion distribution, whose validity can be proved by *weak-law* or *Chebyshev's-inequality* argument.

Lemma (degeneration of normalized Poisson) Let Y_λ be a Poisson random variable with parameter λ, and let $G_\lambda(\cdot)$ be the cdf of a Y_λ/λ. Then

$$\lim_{\lambda\to\infty} G_\lambda(t) = \begin{cases} 1, & \text{if } t > 1; \\ 0, & \text{if } t < 1. \end{cases}$$

Proof: By Chebyshev's inequality,

$$\Pr\left[\left|\frac{Y_\lambda - \lambda}{\lambda}\right| \geq \varepsilon\right] = \Pr\left[|Y_\lambda - \lambda| \geq \varepsilon\lambda\right] \leq \frac{\mathrm{Var}[Y_\lambda]}{\lambda^2\varepsilon^2} = \frac{\lambda}{\lambda^2\varepsilon^2} = \frac{1}{\lambda\varepsilon^2} \to 0$$

as $\lambda \to \infty$. □

Limit of normalized Poisson

Let X be a non-negative random variable.

Derive the *one-sided Laplace transform* of the distribution of X as:

$$M_X(s)_+ = \int_0^\infty e^{-sx} dF_X(x) \text{ for } s \geq 0.$$

Notably, $M_X(s)_+ = \int_0^\infty e^{-sx} dF_X(x) \leq \int_0^\infty dF_X(x) = 1$ is finite for all $s \geq 0$, but may be infinity for $s < 0$.

Here, we are only interested in those s with $s \geq 0$; hence, it is named the *one-sided Laplace transform*.

In addition, $M_X(s)_+ = M_X(-s)$, where $M_X(\cdot)$ is the moment generating function of X.

Application of degeneration of $G_\lambda(t)$ as $\lambda \to \infty$

Proposition Fix a non-negative random variable X. For $y > 0$,

$$\Pr[X \le y] = \lim_{s\to\infty} \sum_{k=0}^{\lfloor sy \rfloor} \frac{(-1)^k}{k!} s^k M_X^{(k)}(s)_+.$$

Proof: For $s > 0$,

$$M_X^{(k)}(s)_+ = (-1)^k \int_0^\infty x^k e^{-sx} dF_X(x).$$

Application of degeneration of $G_\lambda(t)$ as $\lambda \to \infty$

Hence, for $s > 0$,

$$\begin{aligned}
\sum_{k=0}^{\lfloor sy \rfloor} \frac{(-1)^k}{k!} s^k M_X^{(k)}(s)_+ &= \sum_{k=0}^{\lfloor sy \rfloor} \frac{(-1)^k}{k!} s^k \left((-1)^k \int_0^\infty x^k e^{-sx} dF_X(x) \right) \\
&= \int_0^\infty \sum_{k=0}^{\lfloor sy \rfloor} e^{-sx} \frac{(sx)^k}{k!} dF_X(x) \\
&= \int_0^\infty \Pr\left[Y_{sx} \le \lfloor sy \rfloor \right] dF_X(x) \\
&= \int_0^\infty \Pr\left[Y_{sx} \le sy \right] dF_X(x) \\
&= \int_0^\infty \Pr\left[\frac{Y_{sx}}{sx} \le \frac{y}{x} \right] dF_X(x) \\
&= \int_0^\infty G_{sx}\left(\frac{y}{x} \right) dF_X(x).
\end{aligned}$$

Application of degeneration of $G_\lambda(t)$ as $\lambda \to \infty$

As a result,

$$\lim_{s\to\infty}\sum_{k=0}^{\lfloor sy\rfloor}\frac{(-1)^k}{k!}s^k M_X^{(k)}(s)_+ = \lim_{s\to\infty}\int_0^\infty G_{sx}\left(\frac{y}{x}\right)dF_X(x) = \int_0^\infty \lim_{s\to\infty} G_{sx}\left(\frac{y}{x}\right)dF_X(x),$$

since by dominated convergence theorem, $f_n(x) = G_{nx}(y/x) \le 1 = g(x)$ for every n, and $\int_0^\infty g(x)dF_X(x) = 1 < \infty$.

Give a sequence of non-negative μ-measurable function f_n with $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathcal{X}$, except on a subset of $\mathcal{X}$ with μ-measure zero.

Lemma (Fatou's lemma) $\int_{\mathcal{X}}\left[\lim_{n\to\infty} f_n(x)\right]\mu(dx) \le \liminf_{n\to\infty}\int_{\mathcal{X}} f_n(x)\mu(dx)$.

Fatou's lemma indicates that in general, we cannot interchange the order of integration and limit operation.

Theorem (Lebesgue convergence theorem or dominated convergence theorem) If, in addition to non-negativity, $f_n(x) \le g(x)$ for all $x \in \mathcal{X}$, except on a subset of $\mathcal{X}$ with μ-measure zero, and $g(\cdot)$ is μ-integrable in $\mathcal{X}$ (namely, $\int_{\mathcal{X}} g(x)\mu(dx) < \infty$), then $\int_{\mathcal{X}}\left[\lim_{n\to\infty} f_n(x)\right]\mu(dx) = \lim_{n\to\infty}\int_{\mathcal{X}} f_n(x)\mu(dx)$.

Application of degeneration of $G_\lambda(t)$ as $\lambda \to \infty$

Consequently, (for y that has no point mass),

$$\begin{aligned}
\lim_{s\to\infty} \sum_{k=0}^{\lfloor sy \rfloor} \frac{(-1)^k}{k!} s^k M_X^{(k)}(s)_+ &= \int_0^\infty \lim_{s\to\infty} G_{sx}\left(\frac{y}{x}\right) dF_X(x) \\
&= \int_0^y dF_X(x) \\
&= \Pr[X \le y].
\end{aligned}$$

(How to determine $\Pr[X \le y]$ when X has point mass at y? Hint: Right-continuity)

□

Corollary The distribution of a non-negative random variable is uniquely determined by its moment generating function $M_X(s)$ at $s < 0$.

Proof: For $y > 0$,

$$\Pr[X \le y] = \lim_{s\to\infty} \sum_{k=0}^{\lfloor sy \rfloor} \frac{(-1)^k}{k!} s^k \frac{\partial^k M_X(-s)}{\partial s^k}.$$

Determining $\Pr[X = 0]$ by the right-continuity of cdf gives the desired result. □

Final comment: In fact, to determine the cdf of a non-negative random variable X, we only need to know $M_X(s)$ for $s < -s_0$ for any $s_0 > 0$.

Maximal inequalities

The maximal inequalities concern the maxima of partial sums.

Theorem 22.4 (due to Kolmogorov) Suppose that $X_1, X_2, \ldots$ are independent with zero mean and finite variances (not necessarily identically distributed). Then for $\alpha > 0$,

$$\Pr\left[\max_{1\le k\le n}|S_k| \ge \alpha\right] \le \frac{1}{\alpha^2}\mathrm{Var}[S_n],$$

where $S_n = X_1 + \cdots + X_n$.

Chebyshev's inequality said that

$$\Pr[|S_n| \ge \alpha] \le \frac{1}{\alpha^2}\mathrm{Var}[S_n].$$

This theorem strengthens the result that $\alpha^{-2}\mathrm{Var}[S_n]$ not only bounds $\Pr[|S_n| \ge \alpha]$, but also bounds $\Pr\left[\max_{1\le k\le n}|S_k| \ge \alpha\right]$.

Proof: Define the event

$$A_k = \left[|S_1| < \alpha \wedge |S_2| < \alpha \wedge \cdots \wedge |S_{k-1}| < \alpha \wedge |S_k| \ge \alpha\right].$$

Maximal inequalities

Since there is exactly one of $\{A_k\}_{k=1}^{\infty}$ is true,

$$\begin{aligned}
E[S_n^2] &= E\left[S_n^2\left(I_{A_1} + I_{A_2} + \cdots + I_{A_n} + I_{A_{n+1}} + \cdots\right)\right] \\
&\geq E\left[S_n^2\left(I_{A_1} + I_{A_2} + \cdots + I_{A_n}\right)\right] \\
&= \sum_{k=1}^{n} E\left[S_n^2 I_{A_k}\right] \\
&= \sum_{k=1}^{n} E\left[\left(S_k^2 + 2S_k(S_n - S_k) + (S_n - S_k)^2\right) I_{A_k}\right] \\
&\geq \sum_{k=1}^{n} E\left[\left(S_k^2 + 2S_k(S_n - S_k)\right) I_{A_k}\right] \\
&= \sum_{k=1}^{n} E\left[S_k^2 I_{A_k} + 2S_k I_{A_k}(S_n - S_k)\right] \\
&= \sum_{k=1}^{n} \left(E\left[S_k^2 I_{A_k}\right] + 2E\left[S_k I_{A_k}(S_n - S_k)\right]\right) \\
&= \sum_{k=1}^{n} \left(E\left[S_k^2 I_{A_k}\right] + 2E\left[S_k I_{A_k}\right] E\left[S_n - S_k\right]\right),
\end{aligned}$$

where the last step follows from the independence between $S_k I_{A_k}$ and $S_n - S_k$.

Maximal inequalities

Continue the previous derivation:

$$\begin{aligned}
E[S_n^2] &\geq \sum_{k=1}^{n} \left(E\left[S_k^2 I_{A_k}\right] + 2E\left[S_k I_{A_k}\right] E\left[S_n - S_k\right] \right) \\
&= \sum_{k=1}^{n} E\left[S_k^2 I_{A_k}\right] \quad \text{(by the zero mean assumption, } E[S_n - S_k] = 0\text{)} \\
&\geq \sum_{k=1}^{n} E\left[\alpha^2 I_{A_k}\right] \quad (I_{A_k} = 1 \text{ only when } |S_k| \geq \alpha) \\
&= \alpha^2 \sum_{k=1}^{n} \Pr[A_k] \\
&= \alpha^2 \Pr\left[\max_{1\leq k\leq n} |S_k| \geq \alpha\right].
\end{aligned}$$

□

Maximal inequalities

The previous theorem provide a bound for the cdf of $\max_{1\le k\le n}|S_k|$ using the second moment.

We can also bound the cdf of $\max_{1\le k\le n}|S_k|$ by the cdf of $|S_k|$ for $1 \le k \le n$.

Theorem 22.5 (due to Etemadi) Suppose that $X_1, X_2, \ldots$ are independent. For $\alpha \ge 0$,

$$\Pr\left[\max_{1\le k\le n}|S_k| \ge 3\alpha\right] \le 3\max_{1\le k\le n}\Pr[|S_k| \ge \alpha].$$

Proof: Define the event

$$A_k = \left[|S_1| < 3\alpha \wedge |S_2| < 3\alpha \wedge \cdots \wedge |S_{k-1}| < 3\alpha \wedge |S_k| \ge 3\alpha\right].$$

Then

$$\begin{aligned}
\Pr\left[\max_{1\le k\le n}|S_k| \ge 3\alpha\right] &= \Pr\left[\left(\max_{1\le k\le n}|S_k| \ge 3\alpha\right) \wedge (|S_n| \ge \alpha)\right] \\
&\quad + \Pr\left[\left(\max_{1\le k\le n}|S_k| \ge 3\alpha\right) \wedge (|S_n| < \alpha)\right] \\
&\le \Pr\left[|S_n| \ge \alpha\right] + \Pr\left[\left(\max_{1\le k\le n}|S_k| \ge 3\alpha\right) \wedge (|S_n| < \alpha)\right].
\end{aligned}$$

Maximal inequalities

(Continue from the previous slide)

$$
\begin{aligned}
\Pr\left[\max_{1\le k\le n}|S_k|\ge 3\alpha\right] &\le \Pr[|S_n|\ge\alpha]+\Pr\left[\left(\max_{1\le k\le n}|S_k|\ge 3\alpha\right)\wedge(|S_n|<\alpha)\right]\\
&= \Pr[|S_n|\ge\alpha]+\Pr[(A_1\vee A_2\vee\cdots\vee A_n)\wedge(|S_n|<\alpha)]\\
&= \Pr[|S_n|\ge\alpha]+\sum_{k=1}^{n}\Pr[A_k\wedge(|S_n|<\alpha)] \quad (\{A_k\}_{k=1}^n \text{ are disjoint events.})\\
&= \Pr[|S_n|\ge\alpha]+\sum_{k=1}^{n-1}\Pr[A_k\wedge(|S_n|<\alpha)] \quad (\Pr[A_n\wedge(|S_n|<\alpha)]=0)\\
&\le \Pr[|S_n|\ge\alpha]+\sum_{k=1}^{n-1}\Pr[A_k\wedge(|S_n-S_k|>2\alpha)]
\end{aligned}
$$

$$
\begin{aligned}
&|S_n|<\alpha\wedge|S_k|\ge 3\alpha\\
&\Rightarrow (-\alpha<S_n<\alpha\wedge S_k\ge 3\alpha)\vee(-\alpha<S_n<\alpha\wedge S_k\le -3\alpha)\\
&\Rightarrow (S_n<\alpha\wedge -S_k\le -3\alpha)\vee(S_n>-\alpha\wedge -S_k\ge 3\alpha)\\
&\Rightarrow (S_n-S_k<-2\alpha)\vee(S_n-S_k>2\alpha)\\
&\Rightarrow |S_n-S_k|>2\alpha.
\end{aligned}
$$

Maximal inequalities

(Continue from the previous slide)

$$
\begin{aligned}
\Pr\left[\max_{1\le k\le n}|S_k|\ge 3\alpha\right] &\le \Pr\left[|S_n|\ge\alpha\right]+\sum_{k=1}^{n-1}\Pr\left[A_k\wedge(|S_n-S_k|>2\alpha)\right]\\
&= \Pr\left[|S_n|\ge\alpha\right]+\sum_{k=1}^{n-1}\Pr\left[A_k\right]\Pr\left[|S_n-S_k|>2\alpha\right]\\
&\qquad\text{(by the independence of } A_k \text{ and } |S_n-S_k|)\\
&\le \Pr\left[|S_n|\ge\alpha\right]+\max_{1\le k\le n}\Pr\left[|S_n-S_k|\ge 2\alpha\right]\\
&\le \Pr\left[|S_n|\ge\alpha\right]+\max_{1\le k\le n}\Pr\left[|S_n|\ge\alpha\vee|S_k|\ge\alpha\right]\\
&\qquad\text{(Notably, } |x|<\alpha \text{ and } |y|<\alpha \text{ implies } |x-y|<2\alpha.)\\
&\le \Pr\left[|S_n|\ge\alpha\right]+\max_{1\le k\le n}\left(\Pr\left[|S_n|\ge\alpha\right]+\Pr\left[|S_k|\ge\alpha\right]\right)\\
&\le \max_{1\le k\le n}\Pr\left[|S_k|\ge\alpha\right]+\max_{1\le k\le n}\Pr\left[|S_k|\ge\alpha\right]+\max_{1\le k\le n}\Pr\left[|S_k|\ge\alpha\right]\\
&= 3\max_{1\le k\le n}\Pr\left[|S_k|\ge\alpha\right].
\end{aligned}
$$

□

Convergence of $X_1 + X_2 + \cdots + X_n$

Theorem (Implication of Kolmogorov's zero-one law) If $X_1, X_2, \ldots$ are independent binary 0-1 random variables, then $\Pr\left[\sum_{k=1}^{\infty} X_k < \infty\right]$ is either 1 or 0.

Proof:

- Define the event $A_k = [X_k = 1]$. Then $A_1, A_2, \ldots$ are independent events. By the two Borel-Cantelli lemmas, $\Pr[A_n \text{ i.o}]$ is either 1 or 0.

Theorem 4.3 (First Borel-Cantelli lemma)

$$\sum_{n=1}^{\infty} P(A_n) < \infty \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = P(A_n \text{ i.o.}) = 0.$$

Theorem 4.4 (Second Borel-Cantelli Lemma) If $\{A_n\}_{n=1}^{\infty}$ forms an independent sequence of events,

$$\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = P(A_n \text{ i.o.}) = 1.$$

- Apparently, if $A_1, A_2, \ldots$ are valid infinitely often in n with probability 1, $\sum_{k=1}^{n} X_k = \infty$ with probability 1.

Convergence of $X_1 + X_2 + \cdots + X_n$

- On the contrary, if $A_1, A_2, \ldots$ are valid finitely many times in n with probability 1, $\sum_{k=1}^{\infty} X_k < \infty$ with probability 1. □

Theorem (general version) If $X_1, X_2, \ldots$ are independent random variables, then $\Pr\left[\sum_{k=1}^{\infty} X_k < \infty\right]$ is either 1 or 0.

- In general, to determine whether $\sum_{k=1}^{\infty} X_k$ converge or diverge is hard.

- In what follows, we provide theorems that can tell whether $\sum_{k=1}^{\infty} X_k$ converges by their moments.

Theorem 22.6 Suppose that $X_1, X_2, \ldots$ are **pair-wise** independent with zero mean. Then, if $\sum_{k=1}^{\infty} \text{Var}[X_k] < \infty$, $\sum_{k=1}^{\infty} X_k < \infty$ with probability 1.

Proof:

Again, I use a different proof from Billingsley's book, which is easier to understand for engineering-major students. It suffices to prove that $\Pr[\max_{k \geq 1} |S_{n+k}| < \infty] = 1$.

- First, for any n fixed, $|S_n| < \infty$ with probability 1 because it were not true, we have $\Pr[|S_n| = \infty] > 0$. Derive

$$\Pr[|S_n| \geq L] \;\leq\; \frac{1}{L^2} \sum_{k=1}^{n} \text{Var}[X_k]. \text{ (by zero mean and Chebyshev's ineq)}$$

As $\Pr[|S_n| \geq L]$ is non-increasing in L, its limit exists, and

$$\lim_{L \to \infty} \Pr[|S_n| \geq L] \;=\; 0,$$

a contradiction to $\Pr[|S_n| = \infty] > 0$.

Convergence of $X_1 + X_2 + \cdots + X_n$

- Secondly, for any n fixed, $\boxed{\max_{k\ge 1} |S_{n+k} - S_n| < \infty \text{ with probability } 1}$, because if $\Pr[\max_{k\ge 1} |S_{n+k} - S_n| = \infty] > 0$, then a contradiction can be obtained as follows.

$$\begin{aligned}
\Pr\left[\max_{1\le k\le r} |S_{n+k} - S_n| \ge L\right] &\le \frac{1}{L^2}\text{Var}\left[S_{n+r} - S_n\right] \text{ (by Theorem 22.4 on Slide 22-19)} \\
&= \frac{1}{L^2}\text{Var}\left[X_{n+1} + \cdots + X_{n+r}\right] \\
&= \frac{1}{L^2}\sum_{k=1}^{r}\text{Var}[X_{n+k}] \text{ (by pair-wise independence)} \\
&\le \frac{1}{L^2}\sum_{k=1}^{\infty}\text{Var}[X_{n+k}].
\end{aligned}$$

Since $\Pr\left[\max_{1\le k\le r} |S_{n+k} - S_n| \ge L\right]$ is non-decreasing in r, its limit exists by the monotone convergence theorem. Thus,

$$\lim_{r\to\infty} \Pr\left[\max_{1\le k\le r} |S_{n+k} - S_n| \ge L\right] = \Pr\left[\max_{k\ge 1} |S_{n+k} - S_n| \ge L\right] \le \frac{1}{L^2}\sum_{k=1}^{\infty}\text{Var}[X_{n+k}].$$

Then by taking L to infinity, we obtain the same contradiction as the previous item.

Convergence of $X_1 + X_2 + \cdots + X_n$

- Thirdly,

$$\Pr[|S_n| < \infty] = 1 \quad \text{and} \quad \Pr\left[\max_{k\geq 1} |S_{n+k} - S_n| < \infty\right] = 1$$

imply

$$\Pr\left[|S_n| < \infty \wedge \max_{k\geq 1} |S_{n+k} - S_n| < \infty\right] = 1.$$

$$\begin{aligned} \Pr(A) = 1 \text{ and } \Pr(B) = 1 &\Rightarrow \Pr(A \cup B) = 1 \\ &\Rightarrow \Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) = 1. \end{aligned}$$

By

$$\max_{k\geq 1} |S_{n+k}| \leq \max_{k\geq 1} (|S_{n+k} - S_n| + |S_n|) \leq \max_{k\geq 1}(|S_{n+k} - S_n|) + |S_n|,$$

we get:

$$\Pr\left[\max_{k\geq 1} |S_{n+k}| < \infty\right] \geq \Pr\left[|S_n| < \infty \wedge \max_{k\geq 1} |S_{n+k} - S_n| < \infty\right] = 1.$$

$\square$

Example 22.2 The *Rademacher functions* $\{r_n(\omega)\}_{n=1}^{\infty}$ on a unit interval are defined as:

$$r_n(\omega) = \begin{cases} +1, & \text{if } d_n = 1; \\ -1, & \text{if } d_n = 0, \end{cases}$$

where $\omega = .d_1 d_2 d_3 \ldots$ is a number lying in $[0, 1)$.

Let W be uniformly distributed over $[0, 1)$.

Define $R_n = r_n(W)$. Then $\{R_n\}_{n=1}^{n}$ is i.i.d. with uniform marginal.

Also, define $X_n = a_n R_n$, where $\{a_n\}_{n=1}^{\infty}$ is a constant sequence.

As a result,

$$\text{Var}[X_n] \;=\; a_n^2 \text{Var}[R_n] = a_n^2.$$

By Theorem 22.6,

$$\sum_{n=1}^{\infty} \text{Var}[X_n] = \sum_{n=1}^{\infty} a_n^2 < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} X_n < \infty \text{ with probability 1.}$$

□

Convergence of $X_1 + X_2 + \cdots + X_n$

A small note on $S_n = \sum_{k=1}^{n} X_k$:

- If S_n converges with probability 1, then S_n converges to some finite random variable S with probability 1.

When convergence in prob. $\Leftrightarrow$ convergence w.p. 1?

Theorem 22.7 For an independent sequence $\{X_n\}$,

$$\sum_{k=1}^{\infty} X_k \text{ converges with probability 1}$$

if, and only if,

$$\sum_{k=1}^{\infty} X_k \text{ converges in probability.}$$

Proof:

1. $$\sum_{n=1}^{\infty} X_n \text{ converges with probability 1}$$

   implies

   $$\sum_{n=1}^{\infty} X_n \text{ converges in probability}$$

   is a known result.

When convergence in prob. ⇔ convergence w.p. 1?

1. S_n converges with probability 1 if
$$\lim_{n\to\infty} \Pr\left[\max_{k\geq 1} |S_{n+k} - S_n| > \varepsilon\right] = 0.$$
2. That S_n converges to S in probability implies
$$\limsup_{n\to\infty} \Pr\left[|S_n - S| > \varepsilon\right] = 0.$$

2. Suppose S_n converges to S in probability.

   Then from Theorem 22.5 (cf. Slide 22-22),

$$\begin{aligned}
\Pr\left[\max_{1\leq k\leq r} |S_{n+k} - S_n| > 3\varepsilon\right] &\leq 3 \max_{1\leq k\leq r} \Pr\left[|S_{n+k} - S_n| \geq \varepsilon\right] \\
&\leq 3 \max_{1\leq k\leq r} \left(\Pr\left[|S_{n+k} - S| \geq \frac{\varepsilon}{2}\right] + \Pr\left[|S_n - S| \geq \frac{\varepsilon}{2}\right]\right) \\
&= 3 \max_{1\leq k\leq r} \Pr\left[|S_{n+k} - S| \geq \frac{\varepsilon}{2}\right] + 3 \Pr\left[|S_n - S| \geq \frac{\varepsilon}{2}\right] \\
&\leq 3 \max_{k\geq 1} \Pr\left[|S_{n+k} - S| \geq \frac{\varepsilon}{2}\right] + 3 \Pr\left[|S_n - S| \geq \frac{\varepsilon}{2}\right].
\end{aligned}$$

So,

$$
\begin{aligned}
\Pr\left[\max_{k\geq 1}|S_{n+k}-S_n|>3\varepsilon\right] &= \lim_{r\to\infty}\Pr\left[\max_{1\leq k\leq r}|S_{n+k}-S_n|>3\varepsilon\right]\\
&\leq 3\max_{k\geq 1}\Pr\left[|S_{n+k}-S|\geq\frac{\varepsilon}{2}\right]+3\Pr\left[|S_n-S|\geq\frac{\varepsilon}{2}\right],
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\limsup_{n\to\infty}\Pr\left[\max_{k\geq 1}|S_{n+k}-S_n|>3\varepsilon\right]\\
&\leq 3\limsup_{n\to\infty}\max_{k\geq 1}\Pr\left[|S_{n+k}-S|\geq\frac{\varepsilon}{2}\right]+3\limsup_{n\to\infty}\Pr\left[|S_n-S|\geq\frac{\varepsilon}{2}\right]\\
&= 3\lim_{n\to\infty}\sup_{\ell\geq n}\max_{k\geq 1}\Pr\left[|S_{\ell+k}-S|\geq\frac{\varepsilon}{2}\right]+3\limsup_{n\to\infty}\Pr\left[|S_n-S|\geq\frac{\varepsilon}{2}\right]\\
&= 3\lim_{n\to\infty}\sup_{k'\geq n+1}\Pr\left[|S_{k'}-S|\geq\frac{\varepsilon}{2}\right]+3\limsup_{n\to\infty}\Pr\left[|S_n-S|\geq\frac{\varepsilon}{2}\right]\\
&= 3\limsup_{n\to\infty}\Pr\left[|S_n-S|\geq\frac{\varepsilon}{2}\right]+3\limsup_{n\to\infty}\Pr\left[|S_n-S|\geq\frac{\varepsilon}{2}\right]=0.
\end{aligned}
$$

□

Three-series theorem

- Alternative conditions for convergence with probability 1.

Theorem 22.8 (Three-series theorem) Suppose that $\{X_n\}_{n=1}^{\infty}$ is independent. Then

1. If

$$\sum_{n=1}^{\infty} \Pr[|X_n| > c], \quad \sum_{n=1}^{\infty} E[X_n I_{[|X_n| \le c]}], \quad \text{and} \quad \sum_{n=1}^{\infty} \text{Var}[X_n I_{[|X_n| \le c]}]$$

   converges for **some** positive c, then $\sum_{n=1}^{\infty} X_n$ converges with probability 1.

2. If $\sum_{n=1}^{n} X_n$ converges with probability 1, then

$$\sum_{n=1}^{\infty} \Pr[|X_n| > c], \quad \sum_{n=1}^{\infty} E[X_n I_{[|X_n| \le c]}], \quad \text{and} \quad \sum_{n=1}^{\infty} \text{Var}[X_n I_{[|X_n| \le c]}]$$

   converge for **all** positive c.

Proof: Omitted. □

Three-series theorem

Example 22.3 Continue from Example 22.2.
Define $X_n = a_n R_n$, where $\{a_n\}_{n=1}^{\infty}$ is a constant sequence, and $\{R_n\}_{n=1}^{\infty}$ is i.i.d. with $\Pr[R_n = 1] = \Pr[R_n = -1] = 1/2$.

By Theorem 22.6,

$$\sum_{n=1}^{\infty} \mathrm{Var}[X_n] = \sum_{n=1}^{\infty} a_n^2 < \infty \quad \Rightarrow \quad \sum_{n=1}^{\infty} X_n \text{ converges with probability 1.}$$

By Theorem 22.8,

$$\sum_{n=1}^{\infty} X_n \text{ converges with probability 1} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \mathrm{Var}\left[a_n R_n\right] = \sum_{n=1}^{\infty} a_n^2 < \infty.$$

So $\sum_{n=1}^{\infty} X_n$ converges with probability 1 if, and only if, $\sum_{n=1}^{\infty} a_n^2 < \infty$.

Three-series theorem

By Theorem 22.8,

$$\sum_{n=1}^{\infty} X_n \text{ converges with probability 1} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \Pr\left[|a_n R_n| > c\right] = \sum_{n=1}^{\infty} I_{[|a_n|>c]} < \infty.$$

$$\Rightarrow \quad a_n \text{ is bounded infinitely often in } n. \quad \square$$