2017 Final Exam for Stochastic Processes

1.

> **Definition (Periodogram)** The periodogram of a process is defined as
>
> $$\boldsymbol{S}_T(\omega) = \frac{1}{2T}|\boldsymbol{X}_T(\omega)|^2 \qquad \boldsymbol{X}_T(\omega) = \int_{-T}^{T} \boldsymbol{x}(t)e^{-j\omega t}dt.$$

(a) (4%) Given data window $c(t)$ and spectral window $W(\omega)$, write down the formula of the so-called smoothed modified periodogram $\boldsymbol{S}_{T,w}(\omega; c)$.

Hint: Data window $c(t)$ is applied directly to data $\boldsymbol{x}(t)$. Spectral window $W(\omega)$ is applied to modified periodogram $\boldsymbol{S}_T(\omega; c)$

(b) (6%) Based on $C_T(\omega) \triangleq \int_{-T}^{T} c(t)e^{-j\omega t}dt$ and

$$E[\boldsymbol{S}_T(\omega; c)] = \frac{1}{4\pi T}\int_{-\infty}^{\infty} S_{xx}(v)|C_T(\omega - v)|^2 dv = \frac{1}{4\pi T}\int_{-\infty}^{\infty} S_{xx}(\omega - v)|C_T(v)|^2 dv, \quad (1)$$

show that

$$E[\boldsymbol{S}_{T,w}(\omega; c)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(v)\left(\frac{1}{4\pi T}\int_{-\infty}^{\infty} W(y)|C_T(\omega - y - v)|^2 dy\right) dv. \quad (2)$$

Solution.

(a)

$$\boldsymbol{S}_{T,w}(\omega; c) = \frac{1}{2\pi}\int_{-\infty}^{\infty} W(y)\boldsymbol{S}_T(\omega - y; c)dy$$

where

$$\boldsymbol{S}_T(\omega; c) = \frac{1}{2T}\left|\int_{-T}^{T} c(t)\boldsymbol{x}(t)e^{-j\omega t}dt\right|^2.$$

(b)

$$\begin{aligned} E[\boldsymbol{S}_{T,w}(\omega; c)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(y)E[\boldsymbol{S}_T(\omega - y; c)]dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(y)\left(\int_{-\infty}^{\infty} S_{xx}(v)\cdot\frac{1}{4\pi T}|C_T(\omega - y - v)|^2 dv\right) dy \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(v)\left(\frac{1}{4\pi T}\int_{-\infty}^{\infty} W(y)|C_T(\omega - y - v)|^2 dy\right) dv. \end{aligned}$$

2. (a) (8%) Suppose $S_{xx}(w - v) = S_{xx}(\omega) + v^2 \cdot P(\omega)$. Show that subject to

$$\begin{cases} i)\ c(t) \text{ is real and non-negative, and} \\ ii)\ \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} |C_T(v)|^2 dv = \int_{-T}^{T} |c(t)|^2 dt = 2T, \end{cases}$$

the optimal data window $c(t)$ that minimizes the bias $\left|E[\boldsymbol{S}_T(\omega; c)] - S_{xx}(\omega)\right|$ satisfy $c''(t) + \lambda c(t) = 0$ for some constant λ.

Hint: Use the formulas of $E[\boldsymbol{S}_T(\omega; c)]$ in Eq. (1) in Problem 1(b), and note that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} v^2 |C_T(v)|^2 dv = \int_{-T}^{T} |c'(t)|^2 dt.$$

Euler-Lagrange Equation for Single function of single variable with higher derivatives

The stationary values of the functional

$$I[f] = \int_{x_0}^{x_1} \mathcal{L}(x, f, f', \ldots, f^{(k)}) dx$$

can be obtained from the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial f} - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial f'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial \mathcal{L}}{\partial f''}\right) - \cdots + (-1)^k \frac{d^k}{dx^k}\left(\frac{\partial \mathcal{L}}{\partial f^{(k)}}\right) = 0$$

under fixed boundary conditions for the function itself as well as for the first $k-1$ derivatives.

(b) (8%) The solution that satisfy $c''(t) + \lambda c(t) = 0$ subject to constraints $i)$ and $ii)$, and initial condition $c(T) = c(-T) = 0$ is

$$C_T^{\diamond}(\omega) = 4\sqrt{2}\pi T \frac{\cos(T\omega)}{(\pi^2 - 4T^2\omega^2)}.$$

Find the condition on spectral window $W(\omega)$ and Fourier-transformed data window $C_T(\omega)$, under which the bias remains the same as the minimum achieved in (a).

Hint: Compare Eqs. (1) and (2) in Problem 1(b).

(c) (8%) Without data window, i.e.,

$$c_T(t) = \begin{cases} 1, & |t| < T; \\ 0, & \text{otherwise,} \end{cases}$$

show that the spectral window satisfying

$$\int_{-T}^{T}\int_{-T}^{T} w(t_1 - t_2) e^{-jvt_1} e^{jvt_2} dt_1 dt_2 = |C_T^{\diamond}(v)|^2$$

achieves the same minimum bias in (a).

Hint: Use (b) and

$$\begin{aligned} |C_T(v-y)|^2 &= C_T(v-y) C_T^*(v-y) \\ &= \left(\int_{-\infty}^{\infty} c_T(t_1) e^{-j(v-y)t_1} dt_1\right)\left(\int_{-\infty}^{\infty} c_T^*(t_2) e^{j(v-y)t_2} dt_2\right) \\ &= \left(\int_{-T}^{T} e^{-j(v-y)t_1} dt_1\right)\left(\int_{-T}^{T} e^{j(v-y)t_2} dt_2\right). \end{aligned}$$

Solution.

(a)

$$\begin{aligned}
E[\boldsymbol{S}_T(\omega;c)] &= \int_{-\infty}^{\infty} S_{xx}(\omega-v)\cdot\frac{1}{4\pi T}|C_T(v)|^2dv \\
&= \int_{-\infty}^{\infty}\left(S_{xx}(\omega)+v^2P(\omega)\right)\frac{1}{4\pi T}|C_T(v)|^2dv \\
&= S_{xx}(\omega)\int_{-\infty}^{\infty}\frac{1}{4\pi T}|C_T(v)|^2dv + P(\omega)\int_{-\infty}^{\infty}v^2\frac{1}{4\pi T}|C_T(v)|^2dv \\
&= S_{xx}(\omega)+P(\omega)\frac{1}{4\pi T}\int_{-\infty}^{\infty}v^2|C_T(v)|^2dv.
\end{aligned}$$

Thus, by Lagrange multipliers technique, to minimize the bias

$$|E[\boldsymbol{S}_T(\omega;c)]-S_{xx}(\omega)| = \left|P(\omega)\frac{1}{4\pi T}\int_{-\infty}^{\infty}v^2|C_T(v)|^2dv\right| = \frac{|P(\omega)|}{4\pi T}\int_{-\infty}^{\infty}v^2|C_T(v)|^2dv,$$

it suffices to minimize

$$\begin{aligned}
M &\triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty}v^2|C_T(v)|^2dv-\lambda\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}|C_T(v)|^2dv-2T\right) \\
&= \int_{-T}^{T}(c'(t))^2dt-\lambda\left(\int_{-T}^{T}c^2(t)dt-2T\right) \\
&= \int_{-T}^{T}\Big[\underbrace{(c'(t))^2-\lambda c^2(t)}_{\mathcal{L}(t,c,c')}\Big]dt+2\lambda T.
\end{aligned}$$

Euler-Lagrange equation implies that the solution should satisfy

$$\frac{\partial\mathcal{L}}{\partial c}-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial c'}\right) = -2\lambda c(t)-2c''(t)=0.$$

(b) (1) gives

$$E[\boldsymbol{S}_T(\omega;c)] = \frac{1}{4\pi T}\int_{-\infty}^{\infty}S_{xx}(\omega-v)|C_T(v)|^2dv,$$

and setting $v'=\omega-v$ in (2) yields

$$E[\boldsymbol{S}_{T,w}(\omega;c)] = \frac{1}{4\pi T}\int_{-\infty}^{\infty}S_{xx}(w-v')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}W(y)|C_T(v'-y)|^2dy\right)dv'.$$

Thus, as long as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}W(y)|C_T(v-y)|^2dy = |C_T^{\diamond}(v)|^2,$$

the bias remains the same as (a).

(c)

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty} W(y)|C_T(v-y)|^2 dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} W(y)\left(\int_{-\infty}^{\infty} c_T(t_1)e^{-j(v-y)t_1}dt_1 \int_{-\infty}^{\infty} c_T^*(t_2)e^{j(v-y)t_2}dt_2\right) dy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_T(t_1)c_T^*(t_2)e^{-jvt_1}e^{jvt_2}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} W(y)e^{jy(t_1-t_2)}dy\right) dt_1 dt_2 \\
&= \int_{-T}^{T}\int_{-T}^{T} w(t_1-t_2)e^{-jvt_1}e^{jvt_2}dt_1 dt_2
\end{aligned}
$$

where

$$
\begin{aligned}
|C_T(v-y)|^2 &= C_T(v-y)C_T^*(v-y) \\
&= \left(\int_{-\infty}^{\infty} c_T(t_1)e^{-j(v-y)t_1}dt_1\right)\left(\int_{-\infty}^{\infty} c_T^*(t_2)e^{j(v-y)t_2}dt_2\right) \\
&= \left(\int_{-T}^{T} e^{-j(v-y)t_1}dt_1\right)\left(\int_{-T}^{T} e^{j(v-y)t_2}dt_2\right)
\end{aligned}
$$

Recall that the Papoulis spectral window $W(\omega) = \frac{1}{2T}|C_T^{\diamond}(\omega)|^2$ **actually** minimizes the **asymptotic** bias, given as

$$\begin{aligned}\lim_{T\to\infty} E[\boldsymbol{S}_{T,w}(\omega;c)] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(v)\left(\int_{-\infty}^{\infty} W(y)\lim_{T\to\infty}\frac{1}{4\pi T}|C_T(\omega-y-v)|^2 dy\right)dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(v)\left(\int_{-\infty}^{\infty} W(y)\delta(\omega-y-v)dy\right)dv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(v)W(\omega-v)dv,\end{aligned}$$

not the **bias** for **finite** T. Here, we give the "exact condition" that a "minimum bias spectral window" for finite T should satisfy. (I add double quotation marks on "exact condition" to indicate that such a "minimum bias spectral window" may not exist! See the explanation below.) Since

$$\begin{aligned}|C_T(v)|^2 &= C_T(v)C_T^*(v) \\ &= \left(\int_{-\infty}^{\infty} c_T(t_1)e^{-jvt_1}dt_1\right)\left(\int_{-\infty}^{\infty} c_T^*(t_2)e^{jvt_2}dt_2\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_T(t_1)c_T^*(t_2)e^{-jvt_1}e^{jvt_2}dt_1dt_2,\end{aligned}$$

there may exist no spectral window that satisfies

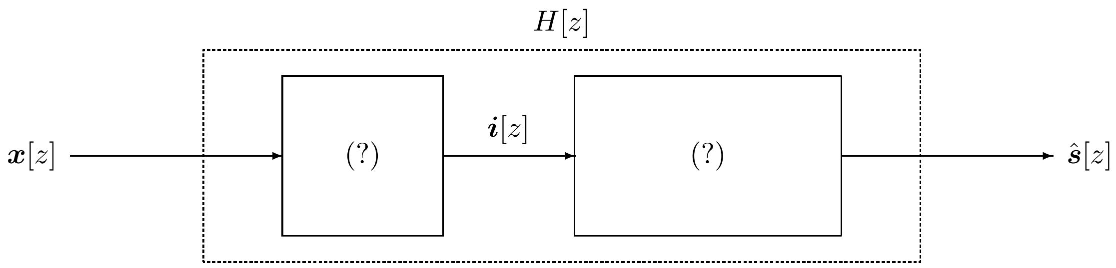
$$\int_{-T}^{T}\int_{-T}^{T} w(t_1-t_2)e^{-jvt_1}e^{jvt_2}dt_1dt_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_T^{\diamond}(t_1)[c_T^{\diamond}(t_2)]^* e^{-jvt_1}e^{jvt_2}dt_1dt_2.$$

This hints that the spectral window cannot achieve the minimum bias that the data window achieves. As a consequence, the spectral window is in general considered more effective for the reduction of the variance of the estimate, rather than the bias!

3. Suppose both $\boldsymbol{s}[n]$ and $\boldsymbol{x}[n]$ are real-valued regular processes, where $\boldsymbol{s}[n] = \sum_{k=0}^{\infty} \mathtt{l}[k]\boldsymbol{i}[n-k]$, $\boldsymbol{x}[n] = \sum_{k=0}^{\infty} \mathtt{m}[k]\boldsymbol{i}[n-k]$, and $\boldsymbol{i}[n]$ is zero-mean real-valued WSS white with unit variance.

   (a) (8%) Prove that the best linear estimator $\hat{\boldsymbol{s}}[n]$ for $\boldsymbol{s}[n]$ based on $\{\boldsymbol{i}[n-k]\}_{k=1}^{\infty}$ is $\hat{\boldsymbol{s}}[n] = \sum_{k=1}^{\infty} \mathtt{l}[k]\boldsymbol{i}[n-k]$.
   Hint: Orthogonality principle.

   (b) (6%) If $\hat{\boldsymbol{s}}[n]$ is the best linear estimator for $\boldsymbol{s}[n]$ based on $\{\boldsymbol{x}[n-k]\}_{k=1}^{\infty}$, then what should be put in the two boxes (with question marks) in the below figure? Here, we respectively use $\boldsymbol{x}[z]$, $\boldsymbol{i}[z]$ and $\boldsymbol{s}[z]$ as the z-transforms of $\boldsymbol{x}[n]$, $\boldsymbol{i}[n]$ and $\boldsymbol{s}[n]$ for notational convenience. Justify your answer.



Hint: Refer to the result in (a).

(c) (8%) Prove that the MS error in (b) is equal to $\mathtt{l}^2[0]$.
Hint: $P = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathtt{E}[e^{j\omega}]|^2 S_{xx}[\omega]d\omega$, where $\mathtt{E}[z]$ is the transfer function of the MS error filter with input $\boldsymbol{x}[n]$ and output $\boldsymbol{e}[n] \triangleq \boldsymbol{s}[n] - \hat{\boldsymbol{s}}[n]$.

(d) (6%) Find the best linear estimator of $\boldsymbol{x}[n]$ based on $\{\hat{\boldsymbol{s}}[n-k]\}_{k=1}^{\infty}$ in (b). What is the minimum MS error in this case?
Hint: $\hat{\boldsymbol{s}}[n]$ is also a regular process.

Solution.

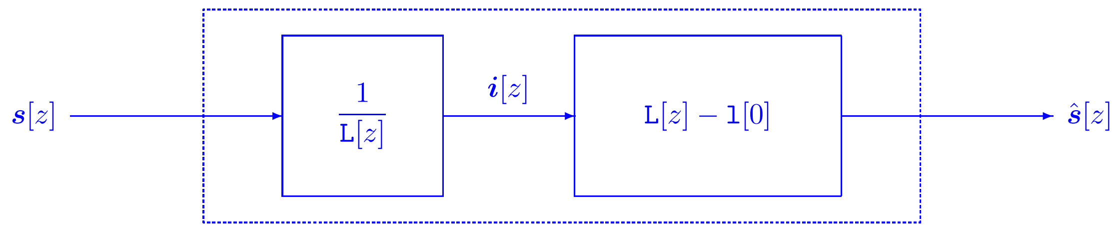
(a) Suppose $\hat{\boldsymbol{s}}[n] = \sum_{k=1}^{\infty} g[k]\boldsymbol{i}[n-k]$ is the optimal linear estimator. Orthogonality principle gives that for all $m \geq 1$,

$$\begin{aligned} 0 &= E\left\{\left(\boldsymbol{s}[n] - \sum_{k=1}^{\infty} g[k]\boldsymbol{i}[n-k]\right)\boldsymbol{i}[n-m]\right\} \\ &= R_{si}[m] - \sum_{k=1}^{\infty} g[k]R_{ii}[m-k] = R_{si}[m] - g[m], \end{aligned}$$

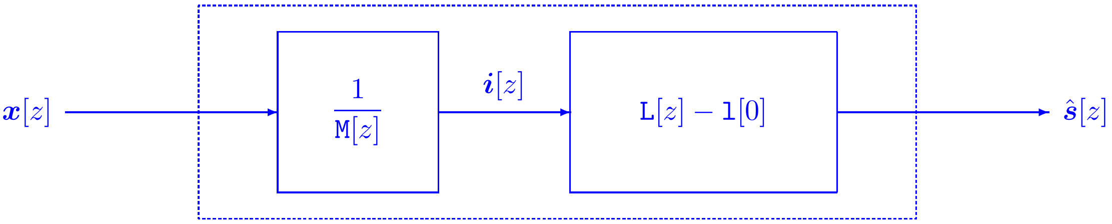
which implies $g[m] = R_{si}[m]$. By regularity, we obtain

$$g[m] = R_{si}[m] = E[\boldsymbol{s}[n]\boldsymbol{i}[n-m]] = \sum_{k=0}^{\infty} \mathtt{l}[k]E\{\boldsymbol{i}[n-k]\boldsymbol{i}[n-m]\} = \mathtt{l}[m].$$

(b) From (a), it is clear that $\hat{\boldsymbol{s}}[z] = (\mathtt{L}[z] - \mathtt{l}[0])\boldsymbol{i}[z]$. In addition, $\boldsymbol{s}[z] = \mathtt{L}[z]\boldsymbol{i}[z]$. Thus, (a) can be graphically illustrated as:



Since $\boldsymbol{x}[z] = \mathtt{M}[z]\boldsymbol{i}[z]$, we can easily see that the best linear estimator $\hat{\boldsymbol{s}}[n]$ based on $\{\boldsymbol{x}[n-k]\}_{k=1}^{\infty}$ should follow:



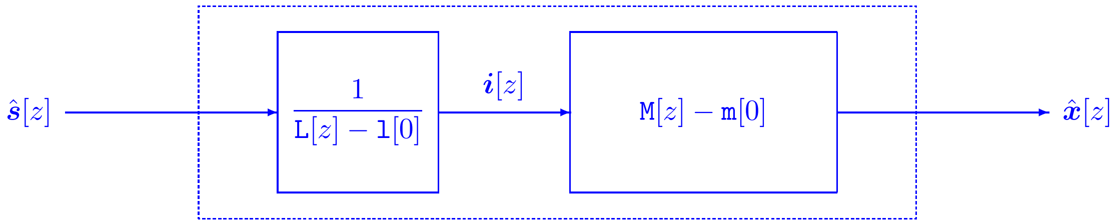
(c) Observing that $\boldsymbol{e}[z] = \boldsymbol{s}[z] - H[z]\boldsymbol{x}[z]$ is the z-transform of the MS error process, we derive

$$\begin{aligned}
\boldsymbol{e}[z] &= \boldsymbol{s}[z] - H[z]\boldsymbol{x}[z] \\
&= \mathtt{L}[z]\boldsymbol{i}[z] - H[z]\boldsymbol{x}[z] \\
&= \frac{\mathtt{L}[z]}{\mathtt{M}[z]}\boldsymbol{x}[z] - \frac{\mathtt{L}[z] - \mathtt{l}[0]}{\mathtt{M}[z]}\boldsymbol{x}[z] \\
&= \frac{\mathtt{l}[0]}{\mathtt{M}[z]}\boldsymbol{x}[z].
\end{aligned}$$

Thus, $\mathtt{E}[z] = \mathtt{l}[0]/\mathtt{M}[z]$ and the MS error is equal to

$$\begin{aligned}
P &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\mathtt{E}[e^{j\omega}]\right|^2 S_{xx}[e^{j\omega}]d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\mathtt{l}[0]}{\mathtt{M}[e^{j\omega}]}\right|^2 \left|\mathtt{M}[e^{j\omega}]\right|^2 d\omega = \mathtt{l}^2[0].
\end{aligned}$$

(d) $\hat{\boldsymbol{s}}[z] = (\mathtt{L}[z] - \mathtt{l}[0])\boldsymbol{i}[z]$ is itself a regular process. Hence, by following (b), the best linear estimator should be of the form:



Observing that

$$\tilde{\boldsymbol{e}}[z] \triangleq \boldsymbol{x}[z] - \hat{\boldsymbol{x}}[z] = \boldsymbol{x}[z] - \frac{\mathtt{M}[z] - \mathtt{m}[0]}{\mathtt{L}[z] - \mathtt{l}[0]}\hat{\boldsymbol{s}}[z]$$

is the z-transform of the MS error process for the above figure, we derive

$$\begin{aligned}
\tilde{\boldsymbol{e}}[z] &= \boldsymbol{x}[z] - \frac{\mathtt{M}[z] - \mathtt{m}[0]}{\mathtt{L}[z] - \mathtt{l}[0]}\hat{\boldsymbol{s}}[z] \\
&= \mathtt{M}[z]\boldsymbol{i}[z] - \frac{\mathtt{M}[z] - \mathtt{m}[0]}{\mathtt{L}[z] - \mathtt{l}[0]}\hat{\boldsymbol{s}}[z] \\
&= \frac{\mathtt{M}[z]}{\mathtt{L}[z] - \mathtt{l}[0]}\hat{\boldsymbol{s}}[z] - \frac{\mathtt{M}[z] - \mathtt{m}[0]}{\mathtt{L}[z] - \mathtt{l}[0]}\hat{\boldsymbol{s}}[z] \\
&= \frac{\mathtt{m}[0]}{\mathtt{L}[z] - \mathtt{l}[0]}\hat{\boldsymbol{s}}[z].
\end{aligned}$$

Thus, $\tilde{\mathtt{E}}[z] = \mathtt{m}[0]/(\mathtt{L}[z] - \mathtt{l}[0])$ and the MS error is equal to

$$\begin{aligned} P &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\tilde{\mathtt{E}}[e^{j\omega}]\right|^2 S_{\hat{s}\hat{s}}[e^{j\omega}]d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\frac{\mathtt{m}[0]}{\mathtt{L}[e^{j\omega}] - \mathtt{l}[0]}\right|^2 \left|\mathtt{L}[e^{j\omega}] - \mathtt{l}[0]\right|^2 d\omega = \mathtt{m}^2[0]. \end{aligned}$$

4. Suppose the best linear r-step-away estimator $\hat{\boldsymbol{s}}_r[n]$ of real-valued random process $\boldsymbol{s}[n]$ based on $\{\boldsymbol{s}[n-k]\}_{k=r}^{\infty}$ is given by $\hat{\boldsymbol{s}}_r[z] = H_r[z]\boldsymbol{s}[z]$, where $\hat{\boldsymbol{s}}_r[z]$ and $\boldsymbol{s}[z]$ are the z-transforms of $\hat{\boldsymbol{s}}_r[n]$ and $\boldsymbol{s}[n]$, respectively. Let $\mathtt{E}_r[z] \triangleq 1 - H_r[z] = \prod_{i=1}^{\infty}(1 - z_i^{(r)} z^{-1})$.

   (a) (8%) Prove that all zeros of $\mathtt{E}_1[z]$ are either inside the unit circle or on the unit circle over the complex z-plane, i.e., $|z_i^{(1)}| \leq 1$ for every $i \geq 1$.
   Hint: Define
   $$\bar{\mathtt{E}}_1[z] = \mathtt{E}_1[z]\frac{1 - z^{-1}/(z_i^{(1)})^*}{1 - z_i^{(1)} z^{-1}}.$$
   Argue using prove-by-contradiction.

   (b) (6%) Is
   $$\bar{\mathtt{E}}_r[z] = \mathtt{E}_r[z]\frac{1 - z^{-1}/(z_i^{(r)})^*}{1 - z_i^{(r)} z^{-1}} = \prod_{k=1}^{\infty}(1 - z_k^{(r)} z^{-1})\frac{1 - z^{-1}/(z_i^{(r)})^*}{1 - z_i^{(r)} z^{-1}}$$
   an MS error filter for an r-step-away linear estimator? Justify your answer.

Solution.

(a) See Slide 13-19.

(b) For an r-step-away linear estimator, $\mathtt{E}_r[z]$ should be of the form:

$$\mathtt{E}_r[z] \;=\; 1 - a_r z^{-r} - a_{r+1} z^{-(r+1)} - \cdots$$

Thus, for $r \geq 2$, the coefficient of z^{-1} should be zero, which implies

$$\sum_{k=1}^{\infty} z_k^{(r)} = 0. \tag{3}$$

Replacing a zero $z_i^{(r)}$ by $1/(z_i^{(r)})^*$ in $\bar{\mathtt{E}}_r[z]$ will change the sum in (3) to:

$$\sum_{k=1, k\neq i}^{\infty} z_k^{(r)} + 1/(z_i^{(r)})^*$$

which is not be equal to zero except when $z_i^{(r)} = 1/(z_i^{(r)})^*$ (or equivalently, $|z_i^{(r)}|^2 = 1$). Thus, $\bar{\mathtt{E}}_r[z]$ is not necessarily an MS error filter for an r-step away linear filter except for $r = 1$.

5. (a) (6%) Suppose $\boldsymbol{s}[n]$ is a real-valued WSS random process. Let $\hat{\boldsymbol{s}}_N[n] = \sum_{k=1}^{N} a_k^{(N)} \boldsymbol{s}[n-k]$ be the best linear estimator of $\boldsymbol{s}[n]$ based on $\{\boldsymbol{s}[n-k]\}_{k=1}^{N}$. Show that the best linear estimator of $\boldsymbol{s}[n-N]$ based on $\{\boldsymbol{s}[(n-N)+k]\}_{k=1}^{N}$ is given by

$$\check{\boldsymbol{s}}_N[n-N] = \sum_{k=1}^{N} a_k^{(N)} \boldsymbol{s}[n-N+k].$$

Hint: Orthogonality principle gives that for $1 \le m \le N$,

$$0 = E\left\{ \left(\boldsymbol{s}[n] - \sum_{k=1}^{N} a_k^{(N)} \boldsymbol{s}[n-k] \right) \boldsymbol{s}[n-m] \right\} = R_{ss}[m] - \sum_{k=1}^{N} a_k^{(N)} R_{ss}[m-k].$$

(b) (6%) Take $N = 1$ in (a). Draw the system diagram with single input $\boldsymbol{s}[n]$ and two outputs $\hat{\boldsymbol{e}}_1[n] \triangleq \boldsymbol{s}[n] - \hat{\boldsymbol{s}}_1[n]$ and $\check{\boldsymbol{e}}_1[n] \triangleq \boldsymbol{s}[n-1] - \check{\boldsymbol{s}}_1[n-1]$, respectively.

(c) (6%) From $\hat{\mathtt{E}}_{N-1}[z] = 1 - a_1^{(N-1)} z^{-1} - \cdots - a_{N-1}^{(N-1)} z^{-(N-1)}$, $\check{\mathtt{E}}_{N-1}[z] = z^{-(N-1)} \hat{\mathtt{E}}_{N-1}[1/z]$, and $\hat{\mathtt{E}}_N[z] = \hat{\mathtt{E}}_{N-1}[z] - k_N z^{-1} \check{\mathtt{E}}_{N-1}[z]$, derive the recursive formula for the computations of $\{a_k^{(N)}\}_{k=1}^{N}$ based on $\{a_k^{(N-1)}\}_{k=1}^{N-1}$ and k_N.
Hint: Comparing termwisely different representations of $\hat{\mathtt{E}}_N[z]$.

(d) (6%) Use the structure in (b) twice to form a system diagram with input $\boldsymbol{s}[n]$ and output $\hat{\boldsymbol{s}}_2[n]$. Please clearly indicate where the input $\boldsymbol{s}[n]$ is and where the output $\hat{\boldsymbol{s}}_2[n]$ is.
Hint: Remember to replace k_1 and k_2 by $a_1^{(1)}$, $a_1^{(2)}$ and $a_2^{(2)}$.

Solution.

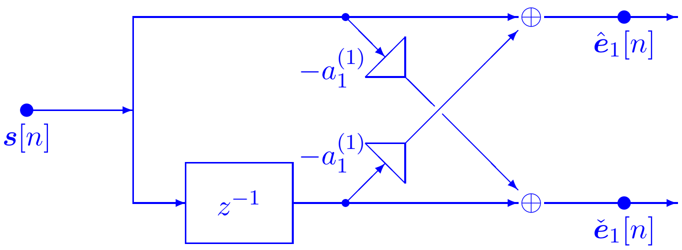
(a) Denote the best linear estimator of $\boldsymbol{s}[n-N]$ based on $\{\boldsymbol{s}[(n-N)+k]\}_{k=1}^{N}$ as $\bar{\boldsymbol{s}}[n-N] = \sum_{k=1}^{N} g[k]\boldsymbol{s}[n-N+k]$. Orthogonality principle implies that for $1 \le m \le N$,

$$\begin{aligned} 0 &= E\left\{ \left(\boldsymbol{s}[n-N] - \sum_{k=1}^{N} g[k]\boldsymbol{s}[n-N+k] \right) \boldsymbol{s}[n-N+m] \right\} \\ &= R_{ss}[m] - \sum_{k=1}^{N} g[k] R_{ss}[k-m]. \end{aligned}$$

Thus, assigning $g[k] = a_k^{(N)}$ must fulfill the above equation and

$$\check{\boldsymbol{s}}_N[n-N] = \sum_{k=1}^{N} a_k^{(N)} \boldsymbol{s}[n-N+k].$$

(b) $\hat{\boldsymbol{s}}_1[n] = a_1^{(1)} \boldsymbol{s}[n-1]$ and $\check{\boldsymbol{s}}_1[n-1] = a_1^{(1)} \boldsymbol{s}[n]$; thus, their MS errors are given by $\hat{\boldsymbol{e}}_1[n] = \boldsymbol{s}[n] - a_1^{(1)} \boldsymbol{s}[n-1]$ and $\check{\boldsymbol{e}}_1[n] = \boldsymbol{s}[n-1] - a_1^{(1)} \boldsymbol{s}[n]$, respectively. The desired system diagram can be illustrated as follows.



(c) See Slide 13-46.

(d) Note that the recursive formula in (c) gives that

$$a_1^{(2)} = a_1^{(1)} - k_2 a_1^{(1)} = (1 - k_2) a_1^{(1)} \quad \text{and} \quad a_2^{(2)} = k_2.$$

Hence, k_2 in the below figure can be marked as either $a_2^{(2)}$ or $1 - a_1^{(2)}/a_1^{(1)}$.