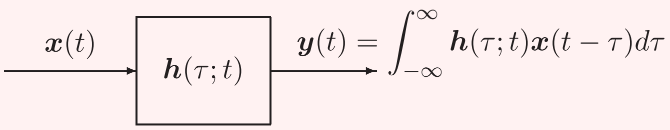
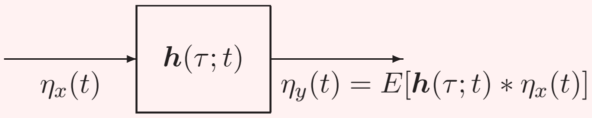
2016 First Midterm for Stochastic Processes

1. Answer the following questions based on Fundamental Theorem and Theorem 9-2.

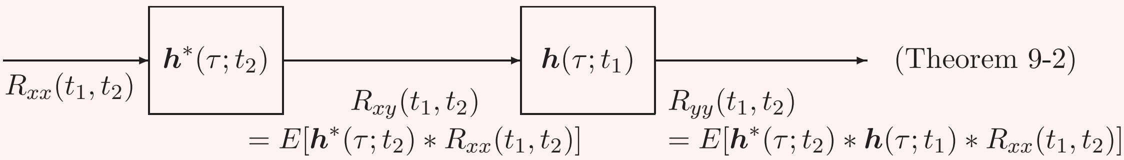
Fundamental Theorem and Theorem 9-2 For any linear system that is defined via convolution operation, i.e.,



we have



(Fundamental Theorem)



(Theorem 9-2)

(a) (6%) Prove that for a (deterministic) linear time-varying system with impulse response $\boldsymbol{h}(\tau; t) = h_1(\tau)e^{j\omega_0 t}$, a zero-mean WSS input always induces a zero-mean WSS output.

(b) (4%) Continue from (a). Under the two premises that

- (i) both the system input $\boldsymbol{x}(t)$ and the system output $\boldsymbol{y}(t)$ are restricted to be zero-mean,
- (ii) the reciprocal of the transfer function $H_1(\omega)$ has inverse Fourier transform,

prove that if the output process $\boldsymbol{y}(t)$ is WSS, then the input process $\boldsymbol{x}(t)$ is WSS, where $H_1(\omega)$ is the Fourier transform of $h_1(\tau)$.

Hint: $\boldsymbol{Y}(\omega) = H(\omega - \omega_0)\boldsymbol{X}(\omega - \omega_0)$.

(c) (4%) Give an impulse response $h_1(\tau)$ that violates the second premise in (b). Show that for your $h_1(\tau)$, a zero-mean non-WSS input process $\boldsymbol{x}(t)$ may induce a zero-mean WSS output $\boldsymbol{y}(t)$.

Solution.

(a) Given that $\boldsymbol{x}(t)$ is zero-mean WSS, we derive

$$
\begin{aligned}
\eta_y(t) &= \int_{-\infty}^{\infty} h(\tau; t)\, \eta_x(t-\tau) d\tau && \text{(By Fundamental Theorem)} \\
&= \int_{-\infty}^{\infty} h(\tau; t)\, \eta_x\, d\tau && \text{(By WSS of } \boldsymbol{x}(t)) \\
&= \eta_x \int_{-\infty}^{\infty} h(\tau; t) d\tau \\
&= 0 && \text{(Due to zero-mean } \boldsymbol{x}(t))
\end{aligned}
$$

and

$$\begin{aligned}
&R_{yy}(t_1,t_2)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(s;t_2)h(\tau;t_1)R_{xx}(t_1-\tau,t_2-s)d\tau ds \quad \text{(By Theorem 9-2)}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(s;t_2)h(\tau;t_1)R_{xx}(t_1-t_2-\tau+s)d\tau ds \quad \text{(By WSS of } \boldsymbol{x}(t))\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1^*(s)e^{-j\omega_0 t_2}h_1(\tau)e^{j\omega_0 t_1}R_{xx}(t_1-t_2-\tau+s)d\tau ds\\
&= e^{j\omega_0(t_1-t_2)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1^*(s)h_1(\tau)R_{xx}((t_1-t_2)-\tau+s)d\tau ds.
\end{aligned}$$

Since $\eta_y(t)$ is equal to zero and $R_{yy}(t_1,t_2)$ is only a function of (t_1-t_2), $\boldsymbol{y}(t)$ is zero-mean WSS.

(b) $\boldsymbol{Y}(\omega) = H(\omega-\omega_0)\boldsymbol{X}(\omega-\omega_0)$ implies that $\boldsymbol{X}(\omega) \triangleq \tilde{H}(\omega+\omega_0)\boldsymbol{Y}(\omega+\omega_0)$, where $\tilde{H}(\omega+\omega_0) = 1/H(\omega)$. Let $\tilde{h}_1(\tau)$ be the inverse Fourier transform of $\tilde{H}_1(\omega)$. Then, $\boldsymbol{x}(t) = \tilde{h}_1(\tau)e^{j(-\omega_0)t} \star \boldsymbol{y}(t)$. By treating $\boldsymbol{x}(t)$ as output and $\boldsymbol{y}(t)$ as input, we can use (a) to confirm that if $\boldsymbol{y}(t)$ is zero-mean WSS, then $\boldsymbol{x}(t)$ must be zero-mean WSS.

> Note: (a) and (b) jointly implies that under $\boldsymbol{h}(\tau;t) = h_1(\tau)e^{j\omega_0 t}$ and the two premises, $\boldsymbol{y}(t)$ is WSS iff $\boldsymbol{x}(t)$ is WSS.

(c) We require that $1/H_1(\omega)$ does not have Fourier transform. A simple example is to have $H_1(\omega)=0$. In such case, $\boldsymbol{y}(t)$ is a zero process for any input process, and thus $\boldsymbol{y}(t)$ is zero-mean WSS for any non-WSS input process.

> Note: Any $H_1(\omega)$ that equals zero over $\omega_1 < \omega < \omega_2$ for some $\omega_1 < \omega_2$ can serve as an example. However, it may be tricky to show how a non-WSS input induces a WSS output for such an $H_1(\omega)$.

2. Suppose $\boldsymbol{x}(t) = \mathrm{Re}\{\boldsymbol{z}(t)\}$, where $\mathrm{Re}\{\cdot\}$ denotes the operation of taking the real part. This is a linear system with input $\boldsymbol{z}(t)$ and output $\boldsymbol{x}(t)$ since it satisfies the superposition principle.

   (a) (6%) Determine the relation between $\eta_z(t)$ and $\eta_x(t)$, where $\eta_z(t)$ and $\eta_x(t)$ are the mean functions of $\boldsymbol{z}(t)$ and $\boldsymbol{x}(t)$, respectively. Can we apply the Fundamental Theorem to the system? Justify your answer.

   Hint: By the system setting, the Fundamental Theorem states that $\eta_x(t) = Re\{\eta_z(t)\}$.

   (b) (6%) Prove that Theorem 9-2 can be applied to the system if, and only if, for any t_1 and t_2,

$$R_{xy}(t_1,t_2) = R_{yy}(t_1,t_2) = 0,$$

   where $\boldsymbol{z}(t) = \boldsymbol{x}(t) + j\boldsymbol{y}(t)$.

   Hint: By the system setting, Theorem 9-2 states that $R_{zx}(t_1,t_2) = Re\{R_{zz}(t_1,t_2)\}$ and $R_{xx}(t_1,t_2) = Re\{R_{zx}(t_1,t_2)\}$. Thus, you should prove that $R_{zx}(t_1,t_2) = Re\{R_{zz}(t_1,t_2)\}$ and $R_{xx}(t_1,t_2) = Re\{R_{zx}(t_1,t_2)\}$ if, and only if, $R_{xy}(t_1,t_2) = R_{yy}(t_1,t_2) = 0$.

   (c) (4%) $R_{yy}(t_1,t_2) = 0$ actually indicates $E[\boldsymbol{y}^2(t)] = 0$, which in turns implies $\boldsymbol{y}(t) = 0$ with probability 1. Thus, Theorem 9-2 can be applied to the linear system, defined by

Re$\{\cdot\}$, only when the input process is trivially real. Infer the reason why Theorem 9-2 can be applied to a linear system like differentiators that have no impulse responses, but in general cannot be applied to Re$\{\cdot\}$ except for the trivial case.

Solution.

(a) $\eta_x(t) = E[\boldsymbol{x}(t)] = E[\text{Re}\{\boldsymbol{z}(t)\}] = \text{Re}\{E[\boldsymbol{z}(t)]\} = \text{Re}\{\eta_z(t)\}$. Since we can obtain $\eta_x(t)$ at the system output by placing $\eta_z(t)$ at the system input, the Fundamental Theorem can be applied to the system.

(b) Derive

$$\begin{aligned} R_{zx}(t_1,t_2) &= E[\boldsymbol{z}(t_1)\boldsymbol{x}^*(t_2)] = E[\boldsymbol{z}(t_1)\boldsymbol{x}(t_2)] \\ &= E\left[\boldsymbol{z}(t_1)\left(\frac{\boldsymbol{z}(t_2)+\boldsymbol{z}^*(t_2)}{2}\right)\right] \\ &= \frac{1}{2}E[\boldsymbol{z}(t_1)\boldsymbol{z}(t_2)] + \frac{1}{2}R_{zz}(t_1,t_2) \end{aligned} \tag{1}$$

and

$$\begin{aligned} R_{xx}(t_1,t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] \\ &= E\left[\left(\frac{\boldsymbol{z}(t_1)+\boldsymbol{z}^*(t_1)}{2}\right)\boldsymbol{x}^*(t_2)\right] \\ &= \frac{1}{2}E\left[\boldsymbol{z}(t_1)\boldsymbol{x}^*(t_2)\right] + \frac{1}{2}E\left[\boldsymbol{z}^*(t_1)\boldsymbol{x}^*(t_2)\right] \\ &= \frac{1}{2}E\left[\boldsymbol{z}(t_1)\boldsymbol{x}^*(t_2)\right] + \frac{1}{2}E\left[\boldsymbol{z}^*(t_1)\boldsymbol{x}(t_2)\right] \quad (\text{By } \boldsymbol{x}^*(t) = \boldsymbol{x}(t)) \\ &= \frac{1}{2}R_{zx}(t_1,t_2) + \frac{1}{2}R^*_{zx}(t_1,t_2) \\ &= \text{Re}\{R_{zx}(t_1,t_2)\}. \end{aligned}$$

Therefore, $R_{xx}(t_1,t_2) = \text{Re}\{R_{zx}(t_1,t_2)\}$ fulfills the need of Theorem 9-2 but $R_{zx}(t_1,t_2) \neq \text{Re}\{R_{zz}(t_1,t_2)\}$. As a result of Eq. (1), Theorem 9-2 can be applied to the system if, and only if, $E[\boldsymbol{z}(t_1)\boldsymbol{z}(t_2)] = R^*_{zz}(t_1,t_2) = E[\boldsymbol{z}^*(t_1)\boldsymbol{z}(t_2)]$, which is equivalent to

$$\begin{aligned} &E[\boldsymbol{z}(t_1)\boldsymbol{z}(t_2)] = E[\boldsymbol{z}^*(t_1)\boldsymbol{z}(t_2)] \\ &\quad\Leftrightarrow\quad E[(\boldsymbol{z}(t_1)-\boldsymbol{z}^*(t_1))\boldsymbol{z}(t_2)] = 0 \\ &\quad\Leftrightarrow\quad E\left[\boldsymbol{y}(t_1)\boldsymbol{z}(t_2)\right] = 0 \\ &\quad\Leftrightarrow\quad E\left[\boldsymbol{x}(t_2)\boldsymbol{y}(t_1)\right] = E\left[\boldsymbol{y}(t_1)\boldsymbol{y}(t_2)\right] = 0 \end{aligned}$$

(c) According to the proofs of Fundamental Theorem and Theorem 9-2, the two theorems holds for all linear systems that have impulse responses. Although a differentiator cannot be defined via a legitimate impulse response, it can be defined via a "transfer function" $(j\omega)$, which can be regarded as the Fourier transform of an "extended impulse response." As a result, the two theorems can still be applied to those linear systems that are defined based on such an "extended impulse response." However, Re$\{\cdot\}$ cannot be defined through an "extended impulse response" (i.e., "transfer function") since

$\boldsymbol{x}(t) = \mathrm{Re}\{\boldsymbol{z}(t)\} = \frac{1}{2}[\boldsymbol{z}(t) + \boldsymbol{z}^*(t)]$ implies $\boldsymbol{X}(\omega) = \frac{1}{2}[\boldsymbol{Z}(\omega) + \boldsymbol{Z}^*(-\omega)]$, and there does not exist a universal $H(\omega)$ to satisfy $\frac{1}{2}[\boldsymbol{Z}(\omega) + \boldsymbol{Z}^*(-\omega)] = H(\omega)\boldsymbol{Z}(\omega)$; thus, Theorem 9-2 cannot be applied to such a linear system.

3. (a) (6%) Suppose $h(\tau;t) = h(\tau)$ is time-invariant (See Theorem 9-2 in Problem 1), and let

$$S_{xx}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2$$

$$S_{xy}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xy}(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2$$

$$S_{yy}(\omega_1,\omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2$$

and

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau.$$

Prove that

$$S_{xy}(\omega_1,\omega_2) = H^*(-\omega_2)S_{xx}(\omega_1,\omega_2) \quad \text{and} \quad S_{yy}(\omega_1,\omega_2) = H(\omega_1)S_{xy}(\omega_1,\omega_2).$$

Hint:

$$\begin{aligned} S_{xy}(\omega_1,\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xy}(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h^*(\tau)R_{xx}(t_1,t_2-\tau)d\tau\right)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2 \end{aligned}$$

and

$$\begin{aligned} S_{yy}(\omega_1,\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(\tau)R_{xy}(t_1-\tau,t_2)d\tau\right)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2 \end{aligned}$$

(b) (6%) Suppose we receive a real process $\boldsymbol{x}(t)$ with autocorrelation function $R_{xx}(t_1,t_2)$ (which is not necessarily WSS). We wish to construct another real process $\boldsymbol{y}(t)$ as the output due to input $\boldsymbol{x}(t)$ and a linear time-invariant filter with real impulse response $h(\tau)$, which satisfies

$$R_{xx}(t_1,t_2) = R_{yy}(t_1,t_2) \quad \text{and} \quad R_{xy}(t_1,t_2) = -R_{yx}(t_1,t_2).$$

Prove that the transfer function of the filter should satisfies that for any ω_1 and ω_2 with $0 < |S_{xx}(\omega_1,\omega_2)| < \infty$,

$$i)\ H^*(-\omega_2)H(\omega_1) = 1 \quad \text{and} \quad ii)\ H(\omega_1) = -H^*(-\omega_2).$$

Hint: Use the results from (a). Here are recapitulations from my slides.

- By Theorem 9-4 (cf. Slide 9-104),

$$S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega) \quad \text{and} \quad S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2.$$

- From $\boldsymbol{X}(\omega) = \boldsymbol{Y}(\omega)[1/H(\omega)]$ and Theorem 9-4 (exchanging the roles of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$), we obtain

$$S_{yx}(\omega) = S_{yy}(\omega)[1/H(\omega)]^* = S_{xx}(\omega)|H(\omega)|^2[1/H(\omega)]^* = S_{xx}(\omega)H(\omega).$$

(c) (4%) Continue from (b). Suppose $0 < |S_{xx}(\omega_1, \omega_2)| < \infty$ for every $\omega_1, \omega_2 \in \Re$. Prove that either 1) $H(\omega) = j$ for every $\omega \in \Re$ or 2) $H(\omega) = -j$ for every $\omega \in \Re$.
Note: Here, $j = \sqrt{-1}$.

(d) (4%) Prove that

$$\boldsymbol{z}(t) = \boldsymbol{x}(t) + j\boldsymbol{y}(t), \quad R_{xx}(t_1, t_2) = R_{yy}(t_1, t_2) \quad \text{and} \quad R_{xy}(t_1, t_2) = -R_{yx}(t_1, t_2)$$

jointly imply that for every $t_1 \neq t_2$,

$$E[\boldsymbol{z}(t_1)\boldsymbol{z}(t_2)] = 0.$$

Solution.

(a)

$$\begin{aligned}
S_{xy}(\omega_1, \omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h^*(\tau)R_{xx}(t_1, t_2 - \tau)d\tau\right)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2 \\
&= \int_{-\infty}^{\infty} h^*(\tau)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1, \underbrace{t_2 - \tau}_{=t_2'})e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1 dt_2\right)d\tau \\
&= \int_{-\infty}^{\infty} h^*(\tau)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1, t_2')e^{-j\omega_1 t_1}e^{-j\omega_2 (t_2'+\tau)}dt_1 dt_2'\right)d\tau \\
&= \int_{-\infty}^{\infty} h^*(\tau)e^{-j\omega_2\tau}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1, t_2')e^{-j\omega_1 t_1}e^{-j\omega_2 t_2'}dt_1 dt_2'\right)d\tau \\
&= S_{xx}(\omega_1, \omega_2)\int_{-\infty}^{\infty} h^*(\tau)e^{-j\omega_2\tau}d\tau \\
&= S_{xx}(\omega_1, \omega_2)\left(\int_{-\infty}^{\infty} h(\tau)e^{-j(-\omega_2)\tau}d\tau\right)^* \\
&= S_{xx}(\omega_1, \omega_2)H^*(-\omega_2),
\end{aligned}$$

$$
\begin{aligned}
S_{yy}(\omega_1,\omega_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t_1,t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1dt_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} h(\tau)R_{xy}(t_1-\tau,t_2)d\tau\right)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1dt_2 \\
&= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xy}(\underbrace{t_1-\tau}_{=t_1'},t_2)e^{-j\omega_1 t_1}e^{-j\omega_2 t_2}dt_1dt_2\right)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xy}(t_1',t_2)e^{-j\omega_1 (t_1'+\tau)}e^{-j\omega_2 t_2}dt_1'dt_2\right)d\tau \\
&= \int_{-\infty}^{\infty} h(\tau)e^{-j\omega_1\tau}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xy}(t_1',t_2)e^{-j\omega_1 t_1'}e^{-j\omega_2 t_2}dt_1'dt_2\right)d\tau \\
&= R_{xy}(\omega_1,\omega_2)H(\omega_1)
\end{aligned}
$$

(b) By (a),

$$S_{xy}(\omega_1,\omega_2) = S_{xx}(\omega_1,\omega_2)H^*(-\omega_2)$$

and

$$S_{yy}(\omega_1,\omega_2) = S_{xx}(\omega_1,\omega_2)H^*(-\omega_2)H(\omega_1).$$

From $\boldsymbol{X}(\omega) = \boldsymbol{Y}(\omega)[1/H(\omega)]$ (exchanging the roles of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$), we obtain again from (a) that

$$S_{yx}(\omega_1,\omega_2) = S_{yy}(\omega_1,\omega_2)[1/H(-\omega_2)]^* = S_{xx}(\omega_1,\omega_2)H(\omega_1).$$

In order to have

$$S_{xx}(\omega_1,\omega_2) = S_{yy}(\omega_1,\omega_2) = S_{xx}(\omega_1,\omega_2)H^*(-\omega_2)H(\omega_1)$$

and

$$S_{xx}(\omega_1,\omega_2)H^*(-\omega_2) = S_{xy}(\omega_1,\omega_2) = -S_{yx}(\omega_1,\omega_2) = -S_{xx}(\omega_1,\omega_2)H(\omega_1),$$

we require that for every ω_1 and ω_2 satisfying that $0 < |S_{xx}(\omega_1,\omega_2)| < \infty$,

$$i)\ H^*(-\omega_2)H(\omega_1) = 1 \quad \text{and} \quad ii)\ H(\omega_1) = -H^*(-\omega_2).$$

(c) The two conditions in (b) jointly imply

$$H^2(\omega_1) = -H^*(-\omega_2)H(\omega_1) = -1,$$

which in turns implies $H(\omega_1) = \pm j$. Taking $H(\omega_1) = j$ into $ii)$ yields that $H(-\omega_2) = j$ for every ω_2 with $0 < |S_{xx}(\omega_1,\omega_2)| < \infty$, i.e., $H(\omega) = j$ for every $\omega \in \Re$. Similarly, taking $H(\omega_1) = -j$ into $ii)$ yields that $H(-\omega_2) = -j$ for every ω_2 with $0 < |S_{xx}(\omega_1,\omega_2)| < \infty$, i.e., $H(\omega) = -j$ for every $\omega \in \Re$.

(d)

$$
\begin{aligned}
E[\boldsymbol{z}(t_1)\boldsymbol{z}(t_2)] &= E[(\boldsymbol{x}(t_1)+j\boldsymbol{y}(t_1))(\boldsymbol{x}(t_2)+j\boldsymbol{y}(t_2))] \\
&= E[\boldsymbol{x}(t_1)\boldsymbol{x}(t_2)] + jE[\boldsymbol{x}(t_1)\boldsymbol{y}(t_2)] + jE[\boldsymbol{y}(t_1)\boldsymbol{x}(t_2)] - E[\boldsymbol{y}(t_1)\boldsymbol{y}(t_2)] \\
&= [R_{xx}(t_1,t_2) - R_{yy}(t_1,t_2)] + j[R_{xy}(t_1,t_2) + R_{yx}(t_1,t_2)] \\
&= 0
\end{aligned}
$$

4. Suppose that $\varphi(t) = \begin{cases} n[0,t), & t \geq 0; \\ n[t,0), & t < 0 \end{cases}$, where

i) $n[t_1, t_2)$ is Gaussian distributed with mean zero and variance $(t_2 - t_1)$,

ii) and $n[t_1, t_2)$ and $n[t_3, t_4)$ are independent if $[t_1, t_2)$ and $[t_3, t_4)$ are non-overlapping intervals.

(a) (6%) Based on the definition below, prove that $\varphi(t)$ is a Gaussian random process.

Definition A random process $\{x(t), t \in \mathcal{I}\}$ is said to be a *Gaussian random process* if all finite collections of samples of the process are Gaussian random vectors.

Hint: What is the joint distribution of $\varphi(s_1)$, ..., $\varphi(s_m)$, $\varphi(t_1)$, ..., $\varphi(t_k)$ for finite m and k under $s_1 < \cdots < s_m \leq 0 < t_1 < \cdots < t_k$?

(b) (6%) Suppose $z(t) = e^{j(\omega_0 t + \lambda\varphi(t) + \varphi_0)}$, where φ_0 is independent of $\varphi(t)$, and $E[e^{j\varphi_0}] = E[e^{j2\varphi_0}] = 0$. Is $z(t)$ WSS? Justify your answer.

(c) (6%) Prove that the time-averaged power spectrum density of $z(t)$ is

$$\bar{S}_{zz}(\omega) = \frac{4\lambda^2}{\lambda^4 + 4(\omega - \omega_0)^2}.$$

(d) (4%) Is $\varphi(t)$ MS continuous? Is $\varphi(t)$ MS differentiable? Justify your answer.

Definition (MS continuity) A process $x(t)$ is called *MS continuous* if

$$\lim_{\epsilon \downarrow 0} E[|x(t+\epsilon) - x(t)|^2] = 0 \text{ for every } t.$$

Definition (MS differentiability) A process $x(t)$ is called *MS differentiable* if

$$\lim_{\epsilon \downarrow 0} E\left[\left|\frac{x(t+\epsilon) - x(t)}{\epsilon} - \left(\lim_{\gamma \downarrow 0} \frac{x(t+\gamma) - x(t)}{\gamma}\right)\right|^2\right] = 0 \text{ for every } t.$$

(e) (4%) Can we approximate $S_{zz}(\omega)$ using the pdf f_{c_ϵ} of $c_\epsilon(0) \triangleq \frac{\varphi(\epsilon) - \varphi(0)}{\epsilon}$ as indicated by Woodward's Theorem? Justify your answer.

Theorem 10-4 (Woodward's Theorem) Let $x(t) = \cos[\omega_0 t + \lambda \int_0^t c(\alpha) d\alpha + \varphi_0]$. If the process $c(t)$ is continuous and SSS with marginal density $f_c(c)$, and also if $c(t) \perp\!\!\!\perp \varphi_0$, and $E[e^{j\varphi_0}] = E[e^{j2\varphi_0}] = 0$, then for large λ,

$$S_{xx}(\omega) \approx \frac{\pi}{2\lambda}\left[f_c\left(\frac{\omega - \omega_0}{\lambda}\right) + f_c\left(\frac{-\omega - \omega_0}{\lambda}\right)\right].$$

(f) (4%) Can we find an optimal demodulation frequency ω_0^* for received signal $x(t) = \mathrm{Re}\{z(t)\}$ such that $E[|w'(t)|^2]$ is minimized, where $w(t) = z(t)e^{-j\omega_0^* t}$ is the demodulated baseband signal (that is obtained using the demodulation frequency ω_0^*)? Justify your answer.

Hint: Is $E[|w'(t)|^2]$ finite?

Solution.

(a) By definition of $\boldsymbol{\varphi}(t)$, we have

$$\begin{bmatrix} \boldsymbol{\varphi}(s_1) \\ \boldsymbol{\varphi}(s_2) \\ \vdots \\ \boldsymbol{\varphi}(s_{m-1}) \\ \boldsymbol{\varphi}(s_m) \\ \boldsymbol{\varphi}(t_1) \\ \boldsymbol{\varphi}(t_2) \\ \vdots \\ \boldsymbol{\varphi}(t_k) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{n}[s_1, s_2) \\ \boldsymbol{n}[s_2, s_3) \\ \vdots \\ \boldsymbol{n}[s_{m-1}, s_m) \\ \boldsymbol{n}[s_m, 0) \\ \boldsymbol{n}[0, t_1) \\ \boldsymbol{n}[t_1, t_2) \\ \vdots \\ \boldsymbol{n}[t_{k-1}, t_k) \end{bmatrix}$$

and hence

$$\begin{bmatrix} \boldsymbol{\varphi}(s_1) & \boldsymbol{\varphi}(s_2) & \cdots & \boldsymbol{\varphi}(s_{m-1}) & \boldsymbol{\varphi}(s_m) & \boldsymbol{\varphi}(t_1) & \boldsymbol{\varphi}(t_2) & \cdots & \boldsymbol{\varphi}(t_k) \end{bmatrix}^T$$

is a linear combination of independent Gaussian random vectors and must be Gaussian distributed. Accordingly, $\boldsymbol{\varphi}(t)$ is a Gaussian random process.

(b) $\mu_z(t) = E[\boldsymbol{z}(t)] = E[e^{j\lambda\boldsymbol{\varphi}(t)}]E[e^{j\boldsymbol{\varphi}_0}]e^{j\omega_0 t} = 0$ is a constant, and

$$\begin{aligned} R_{zz}(t+\tau, t) &= E[\boldsymbol{z}(t+\tau)\boldsymbol{z}^*(t)] \\ &= E\left[e^{j(\omega_0(t+\tau)+\lambda\boldsymbol{\varphi}(t+\tau)+\boldsymbol{\varphi}_0)}e^{-j(\omega_0 t+\lambda\boldsymbol{\varphi}(t)+\boldsymbol{\varphi}_0)}\right] \\ &= e^{j\omega_0\tau}E\left[e^{j\lambda[\boldsymbol{\varphi}(t+\tau)-\boldsymbol{\varphi}(t)]}\right] \\ &= \begin{cases} e^{j\omega_0\tau}E\left[e^{j\lambda\boldsymbol{n}[t,t+\tau)}\right], & \tau \geq 0; \\ e^{j\omega_0\tau}E\left[e^{j\lambda\boldsymbol{n}[t+\tau,t)}\right], & \tau < 0 \end{cases} \\ &= e^{j\omega_0\tau}E\left[e^{j\lambda\boldsymbol{n}[0,|\tau|)}\right] \end{aligned}$$

is only a function of τ. Thus, $\boldsymbol{z}(t)$ is WSS.

(c)

$$R_{zz}(\tau) = e^{j\omega_0\tau}E\left[e^{j\lambda\boldsymbol{n}[0,|\tau|)}\right] = e^{j\omega_0\tau}e^{-\frac{1}{2}|\tau|\lambda^2}$$

implies

$$\begin{aligned}
S_{zz}(\omega) &= \int_{-\infty}^{\infty} e^{j\omega_0\tau} e^{-\frac{1}{2}|\tau|\lambda^2} e^{-j\omega\tau} d\tau \\
&= \int_{0}^{\infty} e^{-\frac{1}{2}\lambda^2\tau} e^{-j(\omega-\omega_0)\tau} d\tau + \int_{-\infty}^{0} e^{\frac{1}{2}\lambda^2\tau} e^{-j(\omega-\omega_0)\tau} d\tau \quad (\tau' = -\tau) \\
&= \int_{0}^{\infty} e^{-(\frac{1}{2}\lambda^2 + j(\omega-\omega_0))\tau} d\tau + \int_{0}^{\infty} e^{-(\frac{1}{2}\lambda^2 - j(\omega-\omega_0))\tau'} d\tau' \\
&= \frac{1}{-(\frac{1}{2}\lambda^2 + j(\omega-\omega_0))} e^{-(\frac{1}{2}\lambda^2 + j(\omega-\omega_0))\tau}\Big|_0^{\infty} \\
&\quad + \frac{1}{-(\frac{1}{2}\lambda^2 - j(\omega-\omega_0))} e^{-(\frac{1}{2}\lambda^2 - j(\omega-\omega_0))\tau'}\Big|_0^{\infty} \\
&= \frac{1}{\frac{1}{2}\lambda^2 + j(\omega-\omega_0)} + \frac{1}{\frac{1}{2}\lambda^2 - j(\omega-\omega_0)} \\
&= \frac{4\lambda^2}{\lambda^4 + 4(\omega-\omega_0)^2}.
\end{aligned}$$

(d) Since $R_{zz}(\tau) = e^{j\omega_0\tau} e^{-\frac{1}{2}|\tau|\lambda^2}$ is continuous but not differentiable at $\tau = 0$, $\boldsymbol{z}(t)$ is MS continuous but not MS differentiable.

(e) $R_{zz}(\tau)$ is not MS differentiable at $\tau = 0$. Thus, Woodward's Theorem cannot be applied. In fact, $\lim_{\epsilon \downarrow 0} f_{\boldsymbol{c}_\epsilon}$ is no longer a probability density function.

For your information, the time-averaged autocorrelation function and the time-averaged power spectrum density of $\boldsymbol{z}_\epsilon(t) = e^{j(\omega_0 t + \lambda t \frac{\boldsymbol{\varphi}(\epsilon) - \boldsymbol{\varphi}(0)}{\epsilon} + \boldsymbol{\varphi}_0)}$ are respectively given by

$$\bar{R}_{z_\epsilon z_\epsilon}(\tau) = e^{j\omega_0\tau} E\left[e^{j\lambda \frac{\boldsymbol{n}[0,\epsilon)}{\epsilon}\tau}\right] = e^{j\omega_0\tau} e^{-\frac{\lambda^2\tau^2}{2\epsilon}}$$

and

$$\bar{S}_{z_\epsilon z_\epsilon} = \frac{\sqrt{2\pi\epsilon}}{\lambda} e^{-\frac{\epsilon(\omega-\omega_0)^2}{2\lambda^2}} = \frac{2\pi}{\lambda} f_{\boldsymbol{c}_\epsilon}\left(\frac{\omega - \omega_0}{\lambda}\right).$$

(f) Observation 1 indicates that $S_{ww}(\omega) = S_{zz}(\omega + \omega_0^*)$. Hence, the problem becomes to minimize

$$\begin{aligned}
M &\triangleq 2\pi E[|\boldsymbol{w}'(t)|^2] = \int_{-\infty}^{\infty} S_{w'w'}(\omega) d\omega \\
&= \int_{-\infty}^{\infty} \omega^2 S_{ww}(\omega) d\omega = \int_{-\infty}^{\infty} (\omega - \omega_0^*)^2 S_{zz}(\omega) d\omega. \qquad (2)
\end{aligned}$$

At the first glance, the optimal ω_0^* that minimizes (2) should satisfy

$$\omega_0^* = \frac{\int_{-\infty}^{\infty} \omega S_{zz}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{zz}(\omega) d\omega}.$$

However, for any ω_0^*,

$$\begin{aligned} M &= \int_{-\infty}^{\infty}(\omega-\omega_0^*)^2 S_{zz}(\omega)d\omega = \int_{-\infty}^{\infty}(\omega-\omega_0^*)^2\left(\frac{4\lambda^2}{\lambda^4+4(\omega-\omega_0)^2}\right)d\omega \\ &= \frac{\lambda^4}{4}\int_{-\infty}^{\infty}\left(\tilde{\omega}+\frac{2}{\lambda^2}(\omega_0-\omega_0^*)\right)^2\left(\frac{2}{1+\tilde{\omega}^2}\right)d\tilde{\omega} \quad \left(\tilde{\omega}=\frac{2}{\lambda^2}(\omega-\omega_0)\right) \\ &= \infty. \end{aligned}$$

Therefore, we cannot find an ω_0^* that minimizes M.

For your information,

$$\begin{aligned} \omega_0^* &= \frac{\int_{-\infty}^{\infty}\omega S_{zz}(\omega)d\omega}{\int_{-\infty}^{\infty}S_{zz}(\omega)d\omega} \\ &= \frac{\int_{-\infty}^{\infty}\omega\left(\frac{4\lambda^2}{\lambda^4+4(\omega-\omega_0)^2}\right)d\omega}{\int_{-\infty}^{\infty}\left(\frac{4\lambda^2}{\lambda^4+4(\omega-\omega_0)^2}\right)d\omega} \quad \left(\tilde{\omega}=\frac{2}{\lambda^2}(\omega-\omega_0)\right) \\ &= \frac{\int_{-\infty}^{\infty}\frac{\frac{\lambda^2}{2}\tilde{\omega}+\omega_0}{1+\tilde{\omega}^2}d\tilde{\omega}}{\int_{-\infty}^{\infty}\frac{1}{1+\tilde{\omega}^2}d\tilde{\omega}} \\ &= \left(\frac{\lambda^2}{4}\right)\frac{\int_{-\infty}^{\infty}\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}d\tilde{\omega}}{\int_{-\infty}^{\infty}\frac{1}{1+\tilde{\omega}^2}d\tilde{\omega}}+\omega_0 \\ &= \left(\frac{\lambda^2}{4}\right)\frac{\int_0^{\infty}\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}d\tilde{\omega}-\int_0^{\infty}\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}d\tilde{\omega}}{\int_{-\infty}^{\infty}\frac{1}{1+\tilde{\omega}^2}d\tilde{\omega}}+\omega_0 \\ &= \text{undefined} \end{aligned}$$

where the last step follows from $\int_{-\infty}^{\infty}\frac{1}{1+\tilde{\omega}^2}d\tilde{\omega}=\pi$ and $\int_{-\infty}^{\infty}\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}d\tilde{\omega}=\int_0^{\infty}\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}d\tilde{\omega}-\int_0^{\infty}\frac{2\tilde{\omega}}{1+\tilde{\omega}^2}d\tilde{\omega}=\infty-\infty=\text{undefined}$.

5. Give a probability space $(S=\{\oplus,\ominus,\otimes,\oslash\},\mathcal{F}=\big\{\emptyset,\{\oplus,\ominus\},\{\otimes,\oslash\},S\big\},P)$, where $P(\{\oplus,\ominus\})=0.6$ and $P(\{\otimes,\oslash\})=0.4$.
   (a) (4%) Is it possible to construct a (non-deterministic) Gaussian random process that is defined over the given probability space? Justify your answer.

   Hint: See the definition of the Gaussian random process in the previous problem.

   (b) (6%) Is it possible to construct a (weakly) white stationary process $\boldsymbol{w}(t)$ over the given probability space? Justify your answer.

   Note: A process $\boldsymbol{w}(t)$ is (weakly) *white* if $\boldsymbol{w}(t_1)$ and $\boldsymbol{w}(t_2)$ are *uncorrelated* for every $t_1 \neq t_2$.

Solution.

(a) By definition, if $\boldsymbol{x}(t)$ is a Gaussian random process, $\boldsymbol{x}(t)$ for any t (a one-dimensional sample) must be a one-dimensional Gaussian random variable. However, $\boldsymbol{x}(t,\zeta)$ can at most map to two real values. Hence, $\boldsymbol{x}(t)$ can only be a degenerated Gaussian with zero variance (by mapping $\boldsymbol{x}(t,\zeta)$ to the same value for every $\zeta \in S$). Thus, the answer to this question is negative!

(b) See the solution for sample problems for the first quiz.

6. (a) (6%) Prove that the autocorrelation function $R_{xx}(t_1, t_2)$ of a random process $\boldsymbol{x}(t)$ is non-negative definite, namely,

$$\sum_i \sum_j a_i a_j^* R_{xx}(t_i, t_j) \geq 0 \quad \text{for any complex } a_i \text{ and } a_j. \tag{3}$$

(b) (4%) Give a function $R(t_1, t_2)$ that cannot be an autocorrelation function of a random process. Justify your answer.

Solution.

(a) See the solution for sample problems for the first quiz.

(b) See Slide 9-121 as one example. You can construct your examples as long as you can justify them.