2017 Second Midterm for Stochastic Processes

1. (a) (4%) Let $\boldsymbol{x}(t)$ be zero-mean WSCS with period T. Sample $\boldsymbol{x}(t)$ at time instance nT_s for integer n, where T_s is the sampling period. For convenience, denote $\boldsymbol{x}_n = \boldsymbol{x}(nT_s)$. Is $\{\boldsymbol{x}_n\}_{n=-\infty}^{\infty}$ also zero-mean WSCS? If your answer is positive, prove it. If your answer is negative, give the necessary and sufficient condition under which $\{\boldsymbol{x}_n\}_{n=-\infty}^{\infty}$ is zero-mean WSCS.

   Hint: $\{\boldsymbol{x}_n\}_{n=-\infty}^{\infty}$ is zero-mean WSCS if, and only if, there exists an integer M such that $R_{xx}[n, m] = R_{xx}[n + M, m + M]$.

   (b) (4%) Define $\boldsymbol{y}_n = \boldsymbol{x}(\boldsymbol{t}_n)$, where $\{\boldsymbol{t}_n\}$ is a strictly sense stationary process. Is $\{\boldsymbol{y}_n\}_{n=-\infty}^{\infty}$ WSCS? If YES, prove it; otherwise, give a counterexample.

Solution.

(a) The answer is negative. To find the required necessary and sufficient condition, we infer that if $\{\boldsymbol{x}_n\}_{n=-\infty}^{\infty}$ is zero-mean WSCS, there exists an integer M such that $R_{xx}[n, m] = R_{xx}[n + M, m + M]$, i.e.,

$$\begin{aligned} R_{xx}(nT_s + MT_s, mT_s + MT_s) &= R_{xx}[n + M, m + M] \\ &= R_{xx}[n, m] = R_{xx}(nT_s, mT_s), \end{aligned}$$

which is valid for every WSCS $\boldsymbol{x}(t)$ with period T when MT_s is a multiple of T. As a consequence, $\{\boldsymbol{x}_n\}_{n=-\infty}^{\infty}$ is zero-mean WSCS if, and only if, T_s/T is a rational number.

(b) Since $\boldsymbol{y}_n$ must have zero mean, we can focus on the required WSCS property for autocorrelation functions. Derive

$$\begin{aligned} R_{yy}[n + M, m + M] &= E[\boldsymbol{y}_{n+M}\boldsymbol{y}^*_{m+M}] \\ &= E[\boldsymbol{x}(\boldsymbol{t}_{n+M})\boldsymbol{x}^*(\boldsymbol{t}_{m+M})] \\ &= E\{\, E[\boldsymbol{x}(s_{n+M})\boldsymbol{x}^*(s_{m+M})|\boldsymbol{t}_{n+M} = s_{n+M}, \boldsymbol{t}_{m+M} = s_{m+M}]\,\} \\ &= E[R_{xx}(\boldsymbol{t}_{n+M}, \boldsymbol{t}_{m+M})] \\ &= E[R_{xx}(\boldsymbol{t}_n, \boldsymbol{t}_m)] \quad \text{(because } \{\boldsymbol{t}_n\} \text{ is SSS)} \\ &= R_{yy}[n, m]; \end{aligned}$$

hence, $\{\boldsymbol{y}_n\}$ is zero-mean WSCS.

2. (a) (4%) What is the output process $\boldsymbol{y}(t)$ corresponding to feeding input process $\boldsymbol{x}(t)$ into linear filter with transfer function $H(\omega) = (1 - pe^{-j\omega T_0})^n$, where $0 < p < 2$ is a real number?

   Hint: You shall represent $\boldsymbol{y}(t)$ as a linear combination of $\boldsymbol{x}(t)$ and $\{\boldsymbol{x}(t - kT_0)\}_{k=1}^{n}$.

   (b) (4%) Continue from (a). Show that $|\omega| < \frac{1}{T_0}\cos^{-1}(p/2)$ implies $|H(\omega)|^2 < 1$.

   (c) (6%) Can we use (b) to replace the multiplicative factor 1/3 in the second condition in Theorem 10-10 (i.e., (*ii*) a number $T_0 < (1/3)\pi/\sigma$) with a larger number? Justify your answer.

Theorem 10-10 Fix (i) a BL process $\boldsymbol{x}(t)$ with bandwidth σ, (ii) a number $T_0 < (1/3)\pi/\sigma$, and (iii) a constant $\epsilon > 0$ arbitrarily small. There exists a (sufficiently large) positive integer n and a set of coefficients $\{a_k\}_{k=1}^n$ such that

$$\lim_{w\to\infty} \frac{1}{2w}\int_{-w}^{w} E\left[\left|\boldsymbol{x}(t) - \sum_{k=1}^{n} a_k \boldsymbol{x}(t-kT_0)\right|^2\right] dt < \epsilon.$$

Solution.

(a) $(1-pe^{-j\omega T_0})^n = \sum_{k=0}^n (-1)^k \binom{n}{k} p^k e^{-jk\omega T_0} = 1 - \sum_{k=1}^n (-1)^k \binom{n}{k} p^k e^{-jk\omega T_0}$. Therefore, $\boldsymbol{y}(t) = \boldsymbol{x}(t) - \sum_{k=1}^n (-1)^k \binom{n}{k} p^k \boldsymbol{x}(t-kT_0)$.

(b)

$$\begin{aligned} |H(\omega)|^2 &= |1-pe^{-j\omega T_0}|^{2n} \\ &= |1-p\cos(\omega T_0) + jp\sin(\omega T_0)|^{2n} \\ &= |1+p^2 - 2p\cos(\omega T_0)|^n. \end{aligned}$$

Thus, when $|\omega| < \frac{1}{T_0}\cos^{-1}(p/2)$, we have $\frac{p}{2} < \cos(\omega T_0) \le 1$, which implies

$$(1-p)^2 \le 1+p^2 - 2p\cos(\omega T_0) < 1 + p^2 - 2p \times \frac{p}{2} = 1.$$

Consequently, $|H(\omega)|^2 \to 0$ as $n \to \infty$.

(c) The answer to the question is positive, which can be justified as follows.

For a given σ, we need to choose T_0 such that $|H(\omega)|^2 \to 0$ as $n\to\infty$ for $|\omega| < \sigma$. From (b), we learn that σ, T_0 and p should satisfy

$$\sigma < \frac{1}{T_0}\cos^{-1}(p/2).$$

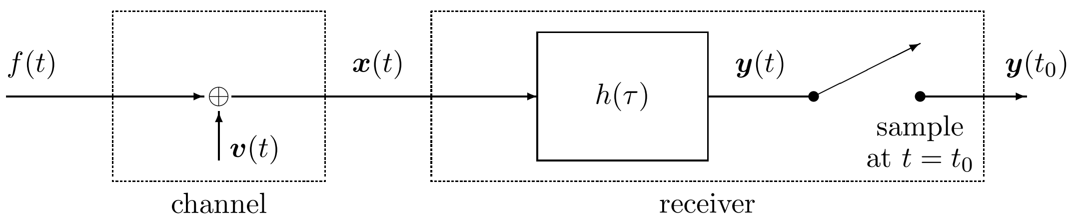
Therefore, the second condition can be generalized to

$$T_0 < \left(\frac{\cos^{-1}(p/2)}{\pi}\right)\frac{\pi}{\sigma}.$$

When $p=1$, $\frac{\cos^{-1}(p/2)}{\pi} = \frac{1}{3}$. Taking any $p<1$ can give a larger $\frac{\cos^{-1}(p/2)}{\pi}$.

3. In the system below, both $f(t)$ and $\boldsymbol{v}(t)$ are real. In addition, $\boldsymbol{v}(t)$ is zero-mean WSS with autocorrelation function $R_{vv}(\tau)$,

$$y_f(t) \triangleq \int_{-\infty}^{\infty} h(\tau) f(t-\tau)d\tau \quad \text{and} \quad \boldsymbol{y}_v(t) \triangleq \int_{-\infty}^{\infty} h(\tau)\boldsymbol{v}(t-\tau)d\tau.$$



(a) (6%) Given that $H(\omega)$ is of the shape:

$$H(\omega) = a_0 + a_m e^{-jm\omega T},$$

find the best real a_0 and a_m such that γ_o is maximized, where

$$R_{vv}(\tau) = \begin{cases} 1, & \tau = 0; \\ 0, & \tau = \pm T, \pm 2T, \ldots \end{cases} \quad \text{and} \quad \gamma_o = \frac{|y_f(t_0)|^2}{E[\boldsymbol{y}_v^2(t_0)]}.$$

(b) (6%) Alternatively, in the above system, find the optimal filter $h(\tau)$ such that $e \triangleq E\{[\boldsymbol{y}(t_0) - f(t_0)]^2\}$ is minimized, provided $f(t) = f(t_0)$ is a constant for $t \in [0, T)$, $R_{vv}(\tau) = \frac{N_0}{2}\delta(\tau)$, and $h(\tau) = 0$ outside $[0, T)$.

Solution.

(a) We can follow similar procedure in Slide 10-69. First note that $y_f(t_0) = a_0 f(t_0) + a_m f(t_0 - mT)$ and $\boldsymbol{y}_v(t_0) = a_0 \boldsymbol{v}(t_0) + a_m \boldsymbol{v}(t - mT)$. Second, to maximize

$$\gamma_o = \frac{|y_f(t_0)|^2}{E[\boldsymbol{y}_v^2(t_0)]} = \frac{c^2}{E[\boldsymbol{y}_v^2(t_0)]}$$

is equivalent to the minimization of $E[\boldsymbol{y}_v^2(t_0)]$ subject to $y_f(t_0) = c$, followed by the maximization with respect to c. Using the Lagrange multiplier technique, we minimize

$$\begin{aligned} V &\triangleq E[\boldsymbol{y}_v^2(t_0)] - 2\lambda(y_f(t_0) - c) \\ &= (a_0^2 + a_m^2) R_{vv}(0) + 2a_0 a_m R_{vv}(mT) - 2\lambda\left(a_0 f(t_0) + a_m f(t_0 - mT) - c\right). \end{aligned}$$

Derive

$$\begin{cases} \dfrac{1}{2}\dfrac{\partial V}{\partial a_0} = a_0 R_{vv}(0) + a_m R_{vv}(mT) - \lambda f(t_0) = 0; \\ \dfrac{1}{2}\dfrac{\partial V}{\partial a_m} = a_m R_{vv}(0) + a_0 R_{vv}(mT) - \lambda f(t_0 - mT) = 0 \end{cases}$$

This leads to:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_m \end{bmatrix} = \lambda \begin{bmatrix} f(t_0) \\ f(t_0 - mT) \end{bmatrix} \triangleq \lambda \vec{f}.$$

As a result, $\vec{a}_{\text{opt}} = \lambda \vec{f}$, and λ is chosen such that $\vec{a}_{\text{opt}}^T \vec{f} = c$, namely,

$$\vec{a}_{\text{opt}}^T \vec{f} = \left(\lambda \vec{f}\right)^T \vec{f} = c \implies \lambda = \frac{c}{\vec{f}^T \vec{f}}.$$

With the availability of the result that

$$\vec{a}_{\text{opt}} = \frac{c}{\vec{f}^T \vec{f}} \vec{f}.$$

we obtain:

$$\gamma_o = \frac{y_f^2(t_0)}{E[\boldsymbol{y}_v^2(t_0)]} = \frac{c^2}{\vec{a}_{\text{opt}}^T \vec{a}_{\text{opt}}} = \frac{c^2}{c^2 \frac{\vec{f}^T \vec{f}}{\left(\vec{f}^T \vec{f}\right)^2}} = \vec{f}^T \vec{f},$$

which is achieved by $\vec{a}_{\text{opt}} = \lambda \vec{f}$ for any $\lambda \neq 0$. (This can be regarded as a tapped-delay-line-structured matched filter.)

(b) We first notice that

$$b = \int_0^T h(\tau) f(t_0) d\tau - f(t_0) = f(t_0) \int_0^T h(\tau) d\tau - f(t_0).$$

By Lagrange multiplier technique, we minimize e subject to

$$\int_0^T h(\tau) d\tau = c,$$

and obtain that for $0 \le v < T$,

$$\begin{aligned} \frac{\partial e}{\partial h(v)} &= \frac{\partial \left[[f(t_0)(c-1)]^2 + \frac{N_0}{2} \int_0^T h^2(\tau) d\tau - \lambda \left(\int_0^T h(\tau) d\tau - c \right) \right]}{\partial h(v)} \\ &= N_0 h(v) - \lambda = 0. \end{aligned}$$

This implies $h_{\text{opt}}(v) = \lambda / N_0$ for $0 \le v < T$.

Alternatively,

$$\begin{aligned} e &= [f(t_0)(c-1)]^2 + \frac{N_0}{2} \int_0^T h^2(\tau) d\tau - \lambda \left(\int_0^T h(\tau) d\tau - c \right) \\ &= \int_0^T \left(\frac{N_0}{2} h^2(\tau) - \lambda h(\tau) \right) d\tau + [f(t_0)(c-1)]^2 + \lambda c \\ &= \int_0^T \left(\frac{N_0}{2} \left[h(\tau) - \frac{\lambda}{N_0} \right]^2 - \frac{\lambda^2}{N_0^2} \right) d\tau + [f(t_0)(c-1)]^2 + \lambda c \end{aligned}$$

implies that taking $h_{\text{opt}}(v) = \lambda / N_0$ minimizes e.

Solving $\int_0^T h_{\text{opt}}(\tau) d\tau = c$ yields $\lambda = c N_0 / T$; and hence, $h_{\text{opt}}(v) = c/T$ for $0 \le v < T$. We continue to derive

$$e = [f(t_0)(c-1)]^2 + \frac{N_0}{2} \int_0^T h^2(\tau) d\tau = [f(t_0)(c-1)]^2 + \frac{N_0 c^2}{2T},$$

which implies e is minimized by taking

$$c_{\text{opt}} = \frac{2 f^2(t_0) T}{2 f^2(t_0) T + N_0}.$$

Consequently, $h_{\text{opt}}(v) = \frac{2 f^2(t_0)}{2 f^2(t_0) T + N_0}$ for $0 \le v < T$, and zero, otherwise.

4. (a) (4%) Let the power spectrum density of a (discrete) WSS process $\boldsymbol{x}[t]$ be

$$S_{xx}[\omega] = \frac{2}{5 - 3\cos(\omega)}.$$

Find its innovation filter $\mathtt{L}[z]$.

Hint: In your answer, $\mathtt{L}[z]$ *should be of the form:*

$$\mathtt{L}[z] = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}},$$

satisfying $S_{xx}[z] = \mathtt{L}[z]\mathtt{L}[z^{-1}]$.

(b) (4%) Find the impulse response $\mathtt{l}[\tau]$ of the innovation filter in (a).

Solution.

(a)

$$S[z] = \frac{4}{10 - 3(z + z^{-1})} = \frac{2/3}{(1 - (1/3)z^{-1})} \cdot \frac{2/3}{(1 - (1/3)z)} = \mathtt{L}[z]\mathtt{L}[z^{-1}]$$

where $\mathtt{L}[z] = \frac{2/3}{1-(1/3)z^{-1}}$.

(b)

$$\begin{aligned} \mathtt{l}[\tau] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathtt{L}[\omega] e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2/3}{(1 - (1/3)e^{-j\omega})} e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2}{3}\left[1 + 3^{-1}e^{-j\omega} + 3^{-2}e^{-j2\omega} + \cdots\right] e^{j\omega\tau} d\omega \\ &= \begin{cases} 0, & \tau < 0 \\ 2 \cdot 3^{-\tau}, & \tau \geq 0 \end{cases} \end{aligned}$$

5. Let the relation of output $\boldsymbol{x}[t]$ and input $\boldsymbol{i}[t]$ be characterized by

$$\boldsymbol{x}[t] + a_1\boldsymbol{x}[t-1] + \cdots + a_n\boldsymbol{x}[t-n] = b_0\boldsymbol{i}[t], \tag{1}$$

where $\boldsymbol{i}[t]$ is real and is a zero-mean white process of unit power (i.e., the autocorrelation function of $\boldsymbol{i}[t]$ satisfies $R_{ii}[\tau] = \delta[\tau]$ with $\delta[\tau]$ being the Kronecker delta function).

(a) (6%) Find the value of $R_{xi}[t-m, t]$ for every $0 \leq m \leq t$.

(b) (6%) Without the premise of $\{\boldsymbol{x}[t]\}$ being WSS, show that the Yule-Walker equations should be generalized to

$$\begin{bmatrix} R_{xx}[t,t] & R_{xx}[t-1,t] & R_{xx}[t-2,t] & \cdots & R_{xx}[t-n,t] \\ R_{xx}[t,t-1] & R_{xx}[t-1,t-1] & R_{xx}[t-2,t-1] & \cdots & R_{xx}[t-n,t-1] \\ R_{xx}[t,t-2] & R_{xx}[t-1,t-2] & R_{xx}[t-2,t-2] & \cdots & R_{xx}[t-n,t-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{xx}[t,t-n] & R_{xx}[t-1,t-n] & R_{xx}[t-2,t-n] & \cdots & R_{xx}[t-n,t-n] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_0^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(c) (4%) Give an example of a_1, a_2, …, a_n such that $\boldsymbol{x}[t]$ is not WSS under $b_0 \neq 0$. You shall provide the proof to justify your example.

(d) (4%) Under $b_0 = 0$ and $n = 1$, (1) is reduced to $\boldsymbol{x}[t] + a_1\boldsymbol{x}[t-1] = 0$. Find the autocorrelation function of $\boldsymbol{x}[t]$ if $a_1 = -e^{j\omega_1}$. Is $\boldsymbol{x}[t]$ WSS when it is zero-mean? Does the autocorrelation function satisfies the Yule-Walker equations in (b)? Justify your answer.

Hint: Find the general relation between $\boldsymbol{x}[t_1]$ and $\boldsymbol{x}[t_2]$ for arbitrary integers t_1 and t_2.

(e) (4%) Under $b_0 = 0$ and $n = 1$, (1) is reduced to $\boldsymbol{x}[t] + a_1\boldsymbol{x}[t-1] = 0$. Find the autocorrelation function of $\boldsymbol{x}[t]$ if $a_1 = -\frac{1}{2}$ and $E[|\boldsymbol{x}[t]|^2] > 0$. Is $\boldsymbol{x}[t]$ WSS when it is zero-mean? Does the autocorrelation function satisfies the Yule-Walker equations in (b)? Justify your answer.

Hint: Find the general relation between $\boldsymbol{x}[t_1]$ and $\boldsymbol{x}[t_2]$ for arbitrary integers t_1 and t_2.

Solution.

(a) Since $\boldsymbol{x}[t-m]$ can be completely determined by

$$\boldsymbol{x}[t-m-1] \text{ upto } \boldsymbol{x}[t-m-n] \text{ and } \boldsymbol{i}[t-m],$$

it only depends on

$$\boldsymbol{i}[t-m], \boldsymbol{i}[t-m-1], \boldsymbol{i}[t-m-2], \ldots.$$

Accordingly, for $m > 0$,

$$R_{xi}[t-m,t] = E\{\boldsymbol{x}[t-m]\boldsymbol{i}[t]\} = E\{\boldsymbol{x}[t-m]\}E\{\boldsymbol{i}[t]\} = 0.$$

For the case of $m = 0$, we obtain by multiplying $\boldsymbol{i}[t]$ followed by taking expectation of both sides of (1) that

$$R_{xi}[t,t] + a_1 \cancel{R_{xi}[t-1,t]} + a_2 \cancel{R_{xi}[t-2,t]} + \cdots + a_n \cancel{R_{xi}[t-n,t]} = R_{xi}[t,t] = b_0.$$

(b) By multiplying $\boldsymbol{x}[t-m]$ for $0 \le m \le n$ followed by taking expectation of both sides of (1), we obtain:

$$\begin{array}{ccccc} \times\boldsymbol{x}[t] & : & R_{xx}[t,t] + a_1 R_{xx}[t-1,t] + \cdots + a_n R_{xx}[t-n,t] & = & b_0^2 \\ \times\boldsymbol{x}[t-1] & : & R_{xx}[t,t-1] + a_1 R_{xx}[t-1,t-1] + \cdots + a_n R_{xx}[t-n,t-1] & = & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \times\boldsymbol{x}[t-n] & : & R_{xx}[t,t-n] + a_1 R_{xx}[t-1,t-n] + \cdots + a_n R_{xx}[t-n,t-n] & = & 0, \end{array}$$

which can be equivalently written to

$$\begin{bmatrix} R_{xx}[t,t] & R_{xx}[t-1,t] & R_{xx}[t-2,t] & \cdots & R_{xx}[t-n,t] \\ R_{xx}[t,t-1] & R_{xx}[t-1,t-1] & R_{xx}[t-2,t-1] & \cdots & R_{xx}[t-n,t-1] \\ R_{xx}[t,t-2] & R_{xx}[t-1,t-2] & R_{xx}[t-2,t-2] & \cdots & R_{xx}[t-n,t-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{xx}[t,t-n] & R_{xx}[t-1,t-n] & R_{xx}[t-2,t-n] & \cdots & R_{xx}[t-n,t-n] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_0^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

(c) A simple example that fulfills the need of the problem is $a_1 = -1$ and $b_0 = 1$, i.e., $\boldsymbol{x}[t] - \boldsymbol{x}[t-1] = \boldsymbol{i}[t]$; hence, $\boldsymbol{x}[t]$ is real.

Suppose $\boldsymbol{x}[t]$ is WSS. Then, the Yule-Walker equations are simplified to

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \\ R_{xx}[1] & R_{xx}[0] \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which implies $R_{xx}[0] - R_{xx}[1] = 1$ and $R_{xx}[1] - R_{xx}[0] = 0$. Since the above equations have no solution, $\boldsymbol{x}[t]$ cannot be WSS.

Since $\boldsymbol{x}[t] = \sum_{k=0}^{\infty} \boldsymbol{i}[t-k]$, we derive for $n \geq m$,

$$\begin{aligned} R_{xx}[n,m] &= E[\boldsymbol{x}[n]\boldsymbol{x}[m]] \\ &= E\left[\left(\sum_{k=0}^{\infty} \boldsymbol{i}[n-k]\right)\left(\sum_{\ell=0}^{\infty} \boldsymbol{i}[m-\ell]\right)\right] \\ &= \sum_{k=0}^{\infty} E\left[\boldsymbol{i}[n-k]\left(\sum_{\ell=0}^{\infty} \boldsymbol{i}[m-\ell]\right)\right] \\ &= \sum_{k=n-m}^{\infty} E\left[\boldsymbol{i}[n-k]\left(\sum_{\ell=0}^{\infty} \boldsymbol{i}[m-\ell]\right)\right] \quad (k' = k-(n-m)) \\ &= \sum_{k'=0}^{\infty} E\left[\boldsymbol{i}[m-k']\left(\sum_{\ell=0}^{\infty} \boldsymbol{i}[m-\ell]\right)\right] = \sum_{k'=0}^{\infty} 1 = \infty. \end{aligned}$$

Thus, the Yule-Walker equations actually imply "$\infty - \infty = 1$" and "$\infty - \infty = 0$", which explains why the derivations in (b) seems convincing but the two equations in this example have no (finite) solution!

Another example that fulfills the need of the problem is $a_1 = -2$ and $b_0 = 1$, i.e., $\boldsymbol{x}[t] - 2\boldsymbol{x}[t-1] = \boldsymbol{i}[t]$; hence, $\boldsymbol{x}[t]$ is real.

Suppose $\boldsymbol{x}[t]$ is WSS. Then, the Yule-Walker equations are simplified to

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \\ R_{xx}[1] & R_{xx}[0] \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

which implies $R_{xx}[0] - 2R_{xx}[1] = 1$ and $R_{xx}[1] - 2R_{xx}[0] = 0$. Solving the equations, we obtain $R_{xx}[0] = -\frac{1}{3}$ and $R_{xx}[1] = -\frac{2}{3}$. Since $R_{xx}[0] = E[|\boldsymbol{x}[t]|^2]$ cannot be negative, $\boldsymbol{x}[t]$ cannot be WSS.

Since $\boldsymbol{x}[t] = \sum_{k=0}^{\infty} 2^k \cdot \boldsymbol{i}[t-k]$, we derive for $n \geq m$,

$$\begin{aligned}
R_{xx}[n,m] &= E[\boldsymbol{x}[n]\boldsymbol{x}[m]] \\
&= E\left[\left(\sum_{k=0}^{\infty} 2^k \cdot \boldsymbol{i}[n-k]\right)\left(\sum_{\ell=0}^{\infty} 2^\ell \cdot \boldsymbol{i}[m-\ell]\right)\right] \\
&= \sum_{k=0}^{\infty} E\left[2^k \cdot \boldsymbol{i}[n-k]\left(\sum_{\ell=0}^{\infty} 2^\ell \cdot \boldsymbol{i}[m-\ell]\right)\right] \\
&= \sum_{k=n-m}^{\infty} 2^k \cdot E\left[\boldsymbol{i}[n-k]\left(\sum_{\ell=0}^{\infty} 2^\ell \cdot \boldsymbol{i}[m-\ell]\right)\right] \quad (k' = k-(n-m)) \\
&= 2^{n-m}\sum_{k'=0}^{\infty} 2^{k'} \cdot E\left[\boldsymbol{i}[m-k']\left(\sum_{\ell=0}^{\infty} \boldsymbol{i}[m-\ell]\right)\right] = 2^{n-m}\sum_{k'=0}^{\infty} 2^{k'} = \infty.
\end{aligned}$$

Thus, the Yule-Walker equations actually imply "$\infty - 2\cdot\infty = 1$" and "$\infty - 2\cdot\infty = 0$", which explains why the derivations in (b) seems convincing but the two equations in this example have "non-justifiable" solution! Note that usually, $\infty - \infty =$ indeterminate; thus, it could be "1" or "0".

(d) Under $a_1 = -e^{j\omega_1}$, we have $\boldsymbol{x}[t_1] = e^{j\omega_1(t_1-t_2)}\boldsymbol{x}[t_2]$ (with probability one) for integers t_1 and t_2. Thus, $|\boldsymbol{x}[t_1]| = |\boldsymbol{x}[t_2]|$ with probability one, which implies $\sigma_x^2 = E[|\boldsymbol{x}[t]|^2]$ is a constant and

$$R_{xx}[t_1,t_2] = E[\boldsymbol{x}[t_1]\boldsymbol{x}^*[t_2]] = e^{j\omega_1(t_1-t_2)}E[|\boldsymbol{x}[t_2]|^2] = \sigma_x^2 e^{j\omega_1(t_1-t_2)}.$$

Since the autocorrelation function is only a function of time difference $(t_1 - t_2)$, the zero-mean $\boldsymbol{x}[t]$ is surely WSS.

In addition, it can be verified that the autocorrelation function in (d) satisfies the Yule-Walker equations in (b). I.e.,

$$\sigma_x^2 \begin{bmatrix} 1 & e^{-j\omega_1} \\ e^{j\omega_1} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -e^{j\omega_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From $\boldsymbol{x}[t_1] = e^{j\omega_1(t_1-t_2)}\boldsymbol{x}[t_2]$, we can deduce

$$\boldsymbol{x}[t] = e^{j\omega_1 t}\boldsymbol{x}[0] = \boldsymbol{c} \cdot e^{j\omega_1 t}$$

by taking $t_1 = t$ and $t_2 = 0$, where $\boldsymbol{c} = \boldsymbol{x}[0]$.

(e) Under $a_1 = -\frac{1}{2}$, we have $\boldsymbol{x}[t_1] = 2^{-(t_1-t_2)}\boldsymbol{x}[t_2]$ (with probability one) for integers t_1 and t_2. Thus,

$$R_{xx}[t_1,t_2] = E[\boldsymbol{x}[t_1]\boldsymbol{x}^*[t_2]] = 2^{-(t_1-t_2)}E[|\boldsymbol{x}[t_2]|^2].$$

Since $E[|\boldsymbol{x}[t]|^2]$ is not a constant (except when $E[|\boldsymbol{x}[t]|^2] = 0$), $R_{xx}[t_1,t_2]$ is not a function of time difference only. Hence, $\boldsymbol{x}[t]$ is not WSS.

In addition, it can be verified that the autocorrelation function in (e) still satisfies the Yule-Walker equations in (b). I.e.,

$$\begin{bmatrix} E[|\boldsymbol{x}[t]|^2] & 2E[|\boldsymbol{x}[t]|^2] \\ \frac{1}{2}E[|\boldsymbol{x}[t-1]|^2] & E[|\boldsymbol{x}[t-1]|^2] \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

When $b_0 = 0$, $\mathtt{L}[z] = \frac{0}{1-(1/2)z^{-1}} = 0$. Thus, it is somewhat tricky to associate it as a "stable LTI filter," not to mention to apply onto it the rule that a WSS input induces a WSS output for a stable LTI filter. In fact, this should be regarded as an "input-less" loop system.

The general time-averaged autocorrelation function $\bar{R}_{xx}[\tau]$ does not exist since

$$\begin{aligned} \bar{R}_{xx}[\tau] &= \lim_{w\to\infty} \frac{1}{2w+1} \sum_{t=-w}^{w} R_{xx}[t+\tau, t] = \lim_{w\to\infty} \frac{1}{2w+1} \sum_{t=-w}^{w} 2^{-\tau} E[|\boldsymbol{x}[t]|^2] \\ &= 2^{-\tau} \lim_{w\to\infty} \frac{1}{2w+1} \sum_{t=-w}^{w} 2^{-2t} E[|\boldsymbol{x}[0]|^2] \\ &= 2^{-\tau} E[|\boldsymbol{x}[0]|^2] \lim_{w\to\infty} \frac{1}{2w+1} \frac{(2^{-2})^{-w}[1-(2^{-2})^{2w+1}]}{[1-(2^{-2})]} = \infty \end{aligned}$$

where in the derivation, we use $E[|\boldsymbol{x}[t_1]|^2] = 2^{-2(t_1-t_2)} E[|\boldsymbol{x}[t_2]|^2]$; however,

$$\begin{aligned} \bar{R}_{xx}[\tau] &= \lim_{w\to\infty} \frac{1}{w} \sum_{t=0}^{w-1} R_{xx}[t+\tau, t] = \lim_{w\to\infty} \frac{1}{w} \sum_{t=0}^{w-1} 2^{-\tau} E[|\boldsymbol{x}[t]|^2] \\ &= 2^{-\tau} \lim_{w\to\infty} \frac{1}{w} \sum_{t=0}^{w-1} 2^{-2t} E[|\boldsymbol{x}[0]|^2] \\ &= 2^{-\tau} E[|\boldsymbol{x}[0]|^2] \lim_{w\to\infty} \frac{1}{w} \cdot \frac{[1-(2^{-2})^{w}]}{[1-(2^{-2})]} = 0. \end{aligned}$$

Note that by definition, the time-averaged autocorrelation function should be irrelevant to the location of time-averaged window taken in its calculation!

6. (a) (6%) Suppose $\omega_0 = 2\pi/T_0$, and

$$\hat{\boldsymbol{x}}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n e^{jn\omega_0 t} \quad \text{and} \quad \boldsymbol{c}_n = \frac{1}{T_0} \int_0^{T_0} \boldsymbol{x}(t) e^{-jn\omega_0 t} dt.$$

Given that $\boldsymbol{x}(t)$ is not WSS, is $E[|\hat{\boldsymbol{x}}(t)|^2] = R_{xx}(t,t)$ for $0 < t < T_0$? Justify your answer.

(b) (4%) For a set of orthonormal functions $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ over $[0, T_0)$, we re-define

$$\hat{\boldsymbol{x}}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \varphi_n(t) \quad \text{and} \quad \boldsymbol{c}_n = \int_0^{T_0} \boldsymbol{x}(t) \varphi_n^*(t) dt.$$

Prove that

$$\int_0^{T_0} \hat{\boldsymbol{x}}(t) \varphi_m^*(t) dt = \int_0^{T_0} \boldsymbol{x}(t) \varphi_m^*(t) dt \quad \text{for every } m.$$

Hint: By orthonormality, we mean that $\int_0^{T_0} \varphi_n(t)\varphi_m^*(t)dt = \begin{cases} 1, & n = m; \\ 0, & \textit{otherwise}. \end{cases}$

(c) (4%) Continue from (b). When $\{\varphi_n(t)\}$ are the eigenfunctions of the autocorrelation function $R_{xx}(t,s)$ of $\boldsymbol{x}(t)$, which satisfies

$$\int_0^{T_0} R_{xx}(t,s)\varphi_n(s)ds = \lambda_n \varphi_n(t),$$

Mercer's Theorem states that:

$$R_{xx}(t,s) = \sum_{n=-\infty}^{\infty} \lambda_n \varphi_n(t)\varphi_n^*(s).$$

Now suppose $\boldsymbol{y}(t)$ is the output process induced by feeding $\boldsymbol{x}(t)$ into a stable linear time-invariant (LTI) filter with impulse response $h(\tau)$. Represent the autocorrelation function $R_{yy}(t,s)$ of $\boldsymbol{y}(t)$ using $\{\phi_n(t)\}$ and $\{\lambda_n\}$, where $\phi_n(t) = \int_{-\infty}^{\infty} h(\tau)\varphi_n(t-\tau)d\tau$.
Hint: Theorem 9-2.

(d) (4%) Continue from (c). Further assume that $\boldsymbol{x}(t)$ is a non-stationary white process with autocorrelation function $R_{xx}(t,s) = q(t)\delta(t-s)$. Find the eigenfunctions and eigenvalues of $R_{xx}(t,s)$ over $(0, T_0)$.
Hint: Start from $\int_0^{T_0} R_{xx}(t,s)\varphi_\lambda(s)ds = \lambda\varphi_\lambda(t)$ *and use the example on Slide 11-75. Here, we adopt the convention that*

$$\int_{-\infty}^{\infty} \delta(s-a)\delta(s-b)ds = \int_{-\infty}^{\infty} \delta(s-a)\delta(a-b)ds = \delta(a-b).$$

Solution.

(a)

$$\begin{aligned}
E[|\hat{\boldsymbol{x}}(t)|^2] &= E\left[\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n e^{jn\omega_0 t}\right)\left(\sum_{m=-\infty}^{\infty} \boldsymbol{c}_m^* e^{-jm\omega_0 t}\right)\right] \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[\boldsymbol{c}_n \boldsymbol{c}_m^*\right] e^{jn\omega_0 t} e^{-jm\omega_0 t} \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \left(\frac{1}{T_0^2}\int_0^{T_0}\int_0^{T_0} E\left[\boldsymbol{x}(u)\boldsymbol{x}^*(v)\right] e^{-jn\omega_0 u} e^{jm\omega_0 v} du dv\right) e^{jn\omega_0 t} e^{-jm\omega_0 t} \\
&= \int_0^{T_0}\int_0^{T_0} R_{xx}(u,v)\left(\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0(t-u)}\right)\left(\frac{1}{T_0}\sum_{m=-\infty}^{\infty} e^{jm\omega_0(v-t)}\right) du dv \\
&= \int_0^{T_0}\int_0^{T_0} R_{xx}(u,v)\left(\sum_{n=-\infty}^{\infty} \delta(t-u+nT_0)\right)\left(\sum_{m=-\infty}^{\infty} \delta(v-t+mT_0)\right) du dv \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \int_0^{T_0}\int_0^{T_0} R_{xx}(u,v)\delta(t-u+nT_0)\delta(v-t+mT_0) du dv \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_{xx}(t+nT_0, t-mT_0)\mathbf{1}\{0<t+nT_0<T_0\}\mathbf{1}\{0<t-mT_0<T_0\} \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_{xx}(t+nT_0, t-mT_0) \\
&\quad \times\mathbf{1}\left\{-\frac{t}{T_0}<n<1-\frac{t}{T_0} \text{ and } \frac{t}{T_0}-1<m<\frac{t}{T_0}\right\} \\
&= R_{xx}(t,t),
\end{aligned}$$

where the last step follows from $0<t<T_0$ (hence, $n=m=0$).

(b)

$$\begin{aligned}
\int_0^{T_0} \hat{\boldsymbol{x}}(t)\varphi_m^*(t)dt &= \int_0^{T_0}\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n\varphi_n(t)\right)\varphi_m^*(t)dt \\
&= \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \int_0^{T_0} \varphi_n(t)\varphi_m^*(t)dt \\
&= \boldsymbol{c}_m = \int_0^{T_0} \boldsymbol{x}(t)\varphi_m^*(t)dt
\end{aligned}$$

(c) From Theorem 9-2,

$$\begin{aligned}
R_{yy}(t,s) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\tau)h(\tau')R_{xx}(t-\tau', s-\tau)d\tau d\tau' \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(\tau)h(\tau') \sum_{n=-\infty}^{\infty} \lambda_n \varphi_n(t-\tau')\varphi_n^*(s-\tau)d\tau d\tau' \\
&= \sum_{n=-\infty}^{\infty} \lambda_n \left(\int_{-\infty}^{\infty} h(\tau')\varphi_n(t-\tau')d\tau'\right)\left(\int_{-\infty}^{\infty} h^*(\tau)\varphi_n^*(s-\tau)d\tau\right) \\
&= \sum_{n=-\infty}^{\infty} \lambda_n \phi_n(t)\phi_n^*(s)
\end{aligned}$$

We can also derive that

$$\begin{aligned}
R_{xy}(t,s) &= \int_{-\infty}^{\infty} h^*(\tau)R_{xx}(t, s-\tau)d\tau \\
&= \int_{-\infty}^{\infty} h^*(\tau)\left(\sum_{n=-\infty}^{\infty} \lambda_n\varphi_n(t)\varphi_n^*(s-\tau)\right)d\tau \\
&= \sum_{n=-\infty}^{\infty} \lambda_n\varphi_n(t)\left(\int_{-\infty}^{\infty} h^*(\tau)\varphi_n^*(s-\tau)d\tau\right) \\
&= \sum_{n=-\infty}^{\infty} \lambda_n\varphi_n(t)\phi_n^*(s)
\end{aligned}$$

Note that $\{\phi_n(t)\}$ are not necessarily orthonormal!

(d) For $t \in (0, T_0)$,

$$\begin{aligned}
&\int_0^{T_0} R_{xx}(t,s)\varphi_\lambda(s)ds = \lambda\varphi_\lambda(t) \\
&\Leftrightarrow \int_0^{T_0} q(t)\delta(t-s)\varphi_\lambda(s)ds = \lambda\varphi_\lambda(t) \\
&\Leftrightarrow q(t)\varphi_\lambda(t) = \lambda\varphi_\lambda(t) \\
&\Leftrightarrow [q(t)-\lambda]\varphi_\lambda(t) = 0
\end{aligned}$$

Suppose $q(t) = \lambda$ only at $t = u \in (0, T_0)$. (There could be other value of t such as $t = v$ that also makes $q(t) = \lambda$. We would treat this case as the eigenvalue λ has several eigenfunctions.) Then, $\varphi_\lambda(t) = \delta(t-u)$ is an eigenfunction corresponding to eigenvalue λ. (If $q(v) = \lambda$ for some $v \neq u$, then $\varphi_\lambda(t) = \delta(t-v)$ is also an eigenfunction corresponding to eigenvalue λ.)

Let's verify $\varphi_\lambda(t)$ as follows:

$$\begin{aligned}\int_0^{T_0} q(t)\delta(t-s)\varphi_\lambda(s)ds &= \int_0^{T_0} q(t)\delta(t-s)\delta(s-u)ds \\ &= q(t)\delta(t-u) = q(u)\delta(t-u) = \lambda\cdot\delta(t-u).\end{aligned}$$

Orthogonality among eigenfunctions is confirmed as follows.

$$\int_0^{T_0} \varphi_\lambda(t)\varphi_{\lambda'}(t)dt = \int_0^{T_0} \delta(t-u)\delta(t-u')dt = \delta(u-u').$$

7. Suppose $\boldsymbol{x}[t]$ is a discrete WSS process. Answer the following questions regarding Wold's decomposition.

   (a) (4%) Suppose $\boldsymbol{x}[t]$ can be decomposed to sum of $\boldsymbol{z}_1[t]$ and $\boldsymbol{z}_2[t]$, where $\boldsymbol{z}_1[t]$ and $\boldsymbol{z}_2[t]$ are orthogonal. Show that if $\boldsymbol{z}_1[t]$ is WSS, then $\boldsymbol{z}_2$ is WSS.

   (b) (4%) In Slide 11-93, we obtain that $E\{\boldsymbol{e}[t+\tau]\boldsymbol{e}^*[t]\} = 0$ for every integer $\tau > 0$. Is $E[|\boldsymbol{e}[t]|^2]$ a constant, independent of t? If your answer is affirmative, give the constant value of $E\{|\boldsymbol{e}[t]|^2\}$. If your answer is negative, give two t_1 and t_2 such that $E\{|\boldsymbol{e}[t_1]|^2\} \neq E\{|\boldsymbol{e}[t_2]|^2\}$.

   (c) (4%) In Slide 11-94, we argue that the error $\boldsymbol{x}_p[t] = \boldsymbol{x}[t] - \boldsymbol{x}_r[t]$ is orthogonal to the data $\boldsymbol{e}[t-k]$ for $k \geq 0$. Show that $\boldsymbol{x}_p[t]$ is actually orthogonal to $\boldsymbol{e}[t-k]$ for every integer k.

   (d) (4%) Wold's decomposition states that a WSS process $\boldsymbol{x}[t]$ can be decomposed into sum of a *regular* process $\boldsymbol{x}_r[t]$ and a *predictable* process $\boldsymbol{x}_p[t]$. Can we obtain $\boldsymbol{x}_r[t]$ as the output of a LTI filter specifically with input $\boldsymbol{x}[t]$? Can we obtain $\boldsymbol{x}_p[t]$ as the output of a LTI filter specifically with input $\boldsymbol{x}[t]$? For each of the two sub-questions in (d), if your answer is affirmative, give the transfer function of the filter? If negative, justify why $\boldsymbol{x}_r[t]$ (respectively, $\boldsymbol{x}_p[t]$) cannot be obtained by feeding $\boldsymbol{x}[t]$ through an LTI filter.

Solution.

(a) By WSS of $\boldsymbol{x}[t]$ and $\boldsymbol{z}_1[t]$, the mean function of $\boldsymbol{z}_2[t]$ is clearly nothing to do with t. Hence, to verify whether $\boldsymbol{z}_2[t]$ is WSS, it suffices to examine the WSS property of the autocorrelation function of $\boldsymbol{z}_2[t]$. By the orthogonality of $\boldsymbol{z}_1[t]$ and $\boldsymbol{z}_2[t]$, we obtain

$$\begin{aligned}R_{xx}[t_1,t_2] &= E\{(\boldsymbol{z}_1[t_1]+\boldsymbol{z}_2[t_1])(\boldsymbol{z}_1[t_2]+\boldsymbol{z}_2[t_2])^*\} \\ &= R_{z_1z_1}[t_1,t_2] + R_{z_2z_2}[t_1,t_2],\end{aligned}$$

which implies

$$R_{z_2z_2}[t_1,t_2] = R_{xx}[t_1-t_2] - R_{z_1z_1}[t_1-t_2].$$

Therefore, the autocorrelation function of $\boldsymbol{z}_2[t]$ is only a function of time difference. This completes the proof of its wide sense stationarity.

(b)

$$\begin{aligned} E\{|\boldsymbol{e}[t]|^2\} = E\{\boldsymbol{e}[t]\boldsymbol{e}^*[t]\} &= E\left\{\boldsymbol{e}[t]\left(\boldsymbol{x}^*[t] - \sum_{k=1}^{\infty} a_k^* \boldsymbol{x}^*[t-k]\right)\right\} \\ &= E\left\{\boldsymbol{e}[t]\boldsymbol{x}^*[t]\right\} \\ &\quad \text{(because } \boldsymbol{e}[t] \text{ is orthogonal to } \boldsymbol{x}[t-k] \text{ for } k \geq 1) \\ &= E\left\{\left(\boldsymbol{x}[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}[t-k]\right)\boldsymbol{x}^*[t]\right\} \\ &= R_{xx}(0) - \sum_{k=1}^{\infty} a_k R_{xx}[k], \end{aligned}$$

which is a constant, independent of t.

(c) It suffices to show that $\boldsymbol{e}[t+m]$ is orthogonal to $\boldsymbol{x}_p[t]$ for $m > 0$. Derive

$$\begin{aligned} \boldsymbol{x}_p[t] &= \boldsymbol{x}[t] - \boldsymbol{x}_r[t] \\ &= \boldsymbol{x}[t] - \sum_{k=0}^{\infty} w_k \boldsymbol{e}[t-k] \\ &= \boldsymbol{x}[t] - \sum_{k=0}^{\infty} w_k \left(\boldsymbol{x}[t-k] - \sum_{n=1}^{\infty} a_k \boldsymbol{x}[t-k-n]\right). \end{aligned}$$

Thus, $\boldsymbol{x}_p[t]$ is a linear combination of $\boldsymbol{x}[t]$ and its past. From Slide 11-92, $\boldsymbol{e}[t+m]$ for $m > 0$ is orthogonal to $\boldsymbol{x}[t]$ and its past. Therefore, $\boldsymbol{e}[t+m]$ is orthogonal to $\boldsymbol{x}_p[t]$ for $m > 0$.

(d) We know that $\boldsymbol{E}[z] = \boldsymbol{X}[z]A[z]$ from Slide 11-95, and $\boldsymbol{X}_r[z] = \boldsymbol{E}[z]W[z]$ from Slide 11-94. Thus, $\boldsymbol{X}_r[z] = \boldsymbol{E}[z]W[z] = \boldsymbol{X}[z]W[z]A[z]$ and we can pass $\boldsymbol{x}[t]$ through filter $A[z]W[z]$ to obtain $\boldsymbol{x}_r[t]$ at the output.

On the other hand, $\boldsymbol{X}_p[z] = \boldsymbol{X}[z] - \boldsymbol{X}_r[z] = (1 - A[z]W[z])\boldsymbol{X}[z]$; hence, we can pass $\boldsymbol{x}[t]$ through filter $(1 - A[z]W[z])$ to obtain $\boldsymbol{x}_p[t]$ at the output.

Note: In order to have a better understanding of Wold's decomposition, let's provide a toy example of $A[z]$ and $W[z]$ for your reference. Suppose $\boldsymbol{x}[t] = \sum_{k=0}^{\infty} \boldsymbol{i}[t-k] + \boldsymbol{c}_0$, where real $\boldsymbol{i}[t]$ is a zero-mean WSS process of unit power, and real $\boldsymbol{c}_0$ is zero-mean with unit variance. Then, the optimal linear predictor of $\boldsymbol{x}[t]$ based on its past is

$$\hat{\boldsymbol{x}}[t] = \boldsymbol{x}[t-1] = \sum_{k=0}^{\infty} \boldsymbol{i}[t-1-k] + \boldsymbol{c}_0,$$

which implies

$$\boldsymbol{x}[t] - \hat{\boldsymbol{x}}[t] = \boldsymbol{i}[t].$$

This can be verified by orthogonality principle as follows:

$$E\{(\boldsymbol{x}[t] - \hat{\boldsymbol{x}}[t])\boldsymbol{x}[t-m]\} = E\left\{\boldsymbol{i}[t]\left(\sum_{k=0}^{\infty} \boldsymbol{i}[t-m-k] + \boldsymbol{c}_0\right)\right\} = 0 \quad \text{for } m \geq 1.$$

As a summary,

$$\boldsymbol{x}_r[t] = \sum_{k=0}^{\infty} \boldsymbol{i}[t-k],\ \ \boldsymbol{x}_p[t] = \boldsymbol{c}_0,\ \ A[z] = 1 - z^{-1} \text{ and } W[z] = \sum_{k=0}^{\infty} z^{-k}.$$

We can then examine what happens in each step of the proof of Wold's decomposition:

- Passing $\boldsymbol{x}[t]$ through $A[z]$ gives $\boldsymbol{i}[t]$, and feeding $\boldsymbol{i}[t]$ into $W[z]$ gives $\boldsymbol{x}_r[t]$. Hence, $\boldsymbol{x}_p[t] = \boldsymbol{c}_0$ is filtered out by $A[z]W[z]$.
- Passing $\boldsymbol{i}[t]$ through $W[z]$ gives $\boldsymbol{x}_r[t]$, and feeding $\boldsymbol{x}_r[t]$ into $A[z]$ gives $\boldsymbol{i}[t]$. Hence, $\boldsymbol{y}[t] = \boldsymbol{i}[t] - \boldsymbol{i}[t] = 0$.