2017 First Quiz for Stochastic Processes

Name _______________ **Student ID** _______________

1. Give a probability space below:

$$(S = \{\oplus, \ominus, \otimes, \oslash\}, \mathcal{F} = \big\{\emptyset, \{\oplus, \ominus\}, \{\otimes, \oslash\}, S\big\}, P),$$

where $P(\{\oplus, \ominus\}) = 0.6$ and $P(\{\otimes, \oslash\}) = 0.4$.

(a) (20%) Construct a **non-deterministic stationary** random process $\{\boldsymbol{x}(t), t \in \Re\}$ that is defined over this probability space. You should justify that the process you define is **non-deterministic** and **stationary**.
Hint: Stationarity implies that

$$P\left(\{\zeta \in S : \boldsymbol{x}(t+c, \zeta) \in A\}\right) = P\left(\{\zeta \in S : \boldsymbol{x}(t, \zeta) \in A\}\right)$$

for every $A \subset \Re$ and for every $t, c \in \Re$.

(b) (20%) Construct a **non-deterministic** but **non-stationary** random process $\boldsymbol{y}(t)$ that is defined over this probability space. You should justify that the process you define is **non-deterministic** and **non-stationary**.

(c) (10%) Is it possible to define a memoryless system $\{\boldsymbol{T}(x), x \in \mathcal{X}\}$ with system input $\boldsymbol{x}(t)$ in (a) and system output $\boldsymbol{y}(t)$ in (b), where $\boldsymbol{T}(x)$ is also defined over the same probability space? Justify your answer with one sentence.

Solution.

(a) For a non-stationary process, we shall have

$$P\left(\{\zeta \in S : \boldsymbol{x}(t+c, \zeta) \in A\}\right) = P\left(\{\zeta \in S : \boldsymbol{x}(t, \zeta) \in A\}\right)$$

for every $A \in \Re$ and for every $t, c \in \Re$. Since the event space is $\mathcal{F} = \big\{\emptyset, \{\oplus, \ominus\}, \{\otimes, \oslash\}, S\big\}$, a random process that is defined over this probability space should satisfy $\boldsymbol{x}(t, \oplus) = \boldsymbol{x}(t, \ominus)$ and $\boldsymbol{x}(t, \otimes) = \boldsymbol{x}(t, \oslash)$ for every t. In addition, "Non-deterministic" requires that $\boldsymbol{x}(t, \oplus) \neq \boldsymbol{x}(t, \otimes)$ for some t. Thus, we may, for convenience, assign $\boldsymbol{x}(t, \oplus) = \boldsymbol{x}(t, \ominus) = 1$ and $\boldsymbol{x}(t, \otimes) = \boldsymbol{x}(t, \oslash) = -1$. This implies

$$\begin{aligned} P\left(\{\zeta \in S : \boldsymbol{x}(t+c, \zeta) \in A\}\right) &= P\left(\{\zeta \in S : \boldsymbol{x}(t, \zeta) \in A\}\right) \\ &= \begin{cases} P(\{\ \}) = 0, & \text{if } \pm 1 \notin A; \\ P(\{\oplus, \ominus\}) = 0.6, & \text{if } 1 \in A \text{ but } -1 \notin A; \\ P(\{\otimes, \oslash\}) = 0.4, & \text{if } -1 \in A \text{ but } 1 \notin A; \\ P(S) = 1, & \text{if } \pm 1 \in A. \end{cases} \end{aligned}$$

As a result (of taking $A = \{1\}$ and $A = \{-1\}$), we must have $\boldsymbol{x}(t+c, \oplus) = \boldsymbol{x}(t+c, \ominus) = 1$ and $\boldsymbol{x}(t+c, \otimes) = \boldsymbol{x}(t+c, \oslash) = -1$ for every $t, c \in \Re$.

(b) Different from (a), we need

$$P\left(\{\zeta \in S : \boldsymbol{y}(t+c,\zeta) \in A\}\right) \neq P\left(\{\zeta \in S : \boldsymbol{y}(t,\zeta) \in A\}\right)$$

for some $A \in \Re$ and for some $t, c \in \Re$. Thus, we may assign

$$\begin{cases} \boldsymbol{y}(t,\oplus) = \boldsymbol{y}(t,\ominus) = 1 \text{ and } \boldsymbol{y}(t,\otimes) = \boldsymbol{y}(t,\oslash) = -1 \text{ for } t < 0; \\ \boldsymbol{y}(t,\oplus) = \boldsymbol{y}(t,\ominus) = -1 \text{ and } \boldsymbol{y}(t,\otimes) = \boldsymbol{y}(t,\oslash) = 1 \text{ for } t \geq 0. \end{cases}$$

This implies that for $t = -\frac{c}{2}$ with $c > 0$ (and $A = \{1\}$),

$$\Pr[\boldsymbol{y}(t+c) = 1] \neq \Pr[\boldsymbol{y}(t) = 1]$$

because

$$\begin{aligned} \Pr[\boldsymbol{y}(t+c) = 1] &= P\left(\{\zeta \in S : \boldsymbol{y}(t+c,\zeta) = 1\}\right) \\ &= P(\{\otimes, \oslash\}) = 0.4 \\ &\neq \Pr[\boldsymbol{y}(t) = 1] \\ &= P\left(\{\zeta \in S : \boldsymbol{y}(t,\zeta) = 1\}\right) \\ &= P(\{\oplus, \ominus\}) = 0.6. \end{aligned}$$

(c) No. We have proved in our lectures that SSS input induces SSS output for memoryless system.

2. (50%) The autocorrelation function $R_{xx}(t_1, t_2)$ of a random process $\boldsymbol{x}(t)$ is non-negative definite, namely,

$$\sum_i \sum_j a_i a_j^* R_{xx}(t_i, t_j) \geq 0 \quad \text{for any complex } a_i \text{ and } a_j. \tag{1}$$

If $\boldsymbol{x}(t)$ is WSS, determine the equivalent condition to (1) using $S_{xx}(\omega)$, where $S_{xx}(\omega)$ is the power spectrum density of $\boldsymbol{x}(t)$.

Solution. See my solution to the sample problems.