Sample problems for the second quiz

1. (a) Find the "extended inverse Fourier transform" of $F(\omega) = 1$ through

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-|\omega|/n} e^{j\omega t} d\omega.$$

   (b) Find the "extended inverse Fourier transform" of $F(\omega) = j\omega$ through

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{-|\omega|/n} e^{j\omega t} d\omega.$$

   (c) Find the Fourier transforms of the extended inverse Fourier transforms in (a) and (b), respectively. Are they equal to 1 and $j\omega$ in (a) and (b), respectively? Please comment on your answers.

Solution.

(a)

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|\omega|/n} e^{j\omega t} d\omega \\
&= \lim_{n\to\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{0} e^{\omega/n} e^{j\omega t} d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{-\omega/n} e^{j\omega t} d\omega \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{2\pi} \int_{0}^{\infty} e^{-\omega'/n} e^{-j\omega' t} d\omega' + \frac{1}{2\pi} \int_{0}^{\infty} e^{-\omega/n} e^{j\omega t} d\omega \right) \quad (\text{Let } \omega' = -\omega) \\
&= \lim_{n\to\infty} \left(\frac{1}{2\pi} \int_{0}^{\infty} e^{-\omega(1/n+jt)} d\omega + \frac{1}{2\pi} \int_{0}^{\infty} e^{-\omega(1/n-jt)} d\omega \right) \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \left\{ \left(-\frac{1}{(1/n+jt)} e^{-\omega(1/n+jt)} \Big|_0^{\infty} \right) + \left(-\frac{1}{(1/n-jt)} e^{-\omega(1/n-jt)} \Big|_0^{\infty} \right) \right\} \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \left\{ \frac{1}{(1/n+jt)} + \frac{1}{(1/n-jt)} \right\} \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \left(\frac{2/n}{1/n^2+t^2} \right) \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \left(\frac{2n}{1+n^2t^2} \right) \\
&= \delta(t),
\end{aligned}$$

where with $s = nt$,

$$\int_{-\infty}^{\infty} \frac{2n}{1+n^2t^2} dt = \int_{-\infty}^{\infty} \frac{2}{1+s^2} ds = 4 \int_{0}^{\infty} \frac{1}{1+s^2} ds = 4\tan^{-1}(s)\Big|_0^{\infty} = 2\pi.$$

(b)

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)e^{-|\omega|/n}e^{j\omega t}d\omega \\
&= \lim_{n\to\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{0}(j\omega)e^{\omega/n}e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{\infty}(j\omega)e^{-\omega/n}e^{j\omega t}d\omega\right) \quad (\text{Let } \omega' = -\omega) \\
&= \lim_{n\to\infty}\left(\frac{1}{2\pi}\int_{0}^{\infty}(-j\omega')e^{-\omega'/n}e^{-j\omega' t}d\omega' + \frac{1}{2\pi}\int_{0}^{\infty}(j\omega)e^{-\omega/n}e^{j\omega t}d\omega\right) \\
&= \lim_{n\to\infty}\left(-\frac{j}{2\pi}\int_{0}^{\infty}\omega e^{-\omega(1/n+jt)}d\omega + \frac{j}{2\pi}\int_{0}^{\infty}\omega e^{-\omega(1/n-jt)}d\omega\right) \\
&= \lim_{n\to\infty}\left\{-\frac{j}{2\pi}\left(-\frac{1}{(1/n+jt)}\left(\omega+\frac{1}{(1/n+jt)}\right)e^{-\omega(1/n+jt)}\Big|_{0}^{\infty}\right)\right. \\
&\qquad \left. +\frac{j}{2\pi}\left(-\frac{1}{(1/n-jt)}\left(\omega+\frac{1}{(1/n-jt)}\right)e^{-\omega(1/n-jt)}\Big|_{0}^{\infty}\right)\right\} \\
&= \lim_{n\to\infty}\frac{j}{2\pi}\left\{\frac{1}{(1/n+jt)^2}-\frac{1}{(1/n-jt)^2}\right\} \\
&= \lim_{n\to\infty}\frac{4t/n}{2\pi(1/n^2+t^2)^2} \\
&= \lim_{n\to\infty}\frac{2tn^3}{\pi(1+n^2t^2)^2} \\
&= 0.
\end{aligned}
$$

(c) By Replication Property of $\delta(t)$,

$$\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}dt = 1,$$

which is consistent to what has been given in (a). However, the Fourier transform of the zero function obtained in (b) is apparently zero, and is not equal to $j\omega$. Therefore, the so-defined extended Fourier transform and extended inverse Fourier transform do not always form a (Fourier) transform pair. As a result, such an extension is a case-by-case applicative rule and hence may not be considered as a theoretically rigorous extension of the conventional Fourier transform.

2. (a) Show that $\boldsymbol{x}(t) = \sum_{n=-\infty}^{\infty}\boldsymbol{c}_n g(t-nT)$ is wide-sense cyclostationary if $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ is zero-mean wide-sense stationary (i.e., $R_{cc}[n,m] = R_{cc}[n-m]$).

Hint: A random process $\boldsymbol{x}(t)$ *is* wide-sense cyclostationary stationary *(WSCS) with period* T *if* $\eta_{xx}(t+mT) = \eta_{xx}(t)$ *and* $R_{xx}(t_1+mT, t_2+mT) = R_{xx}(t_1,t_2)$ *for every integer* m.

(b) Is $\boldsymbol{x}(t)$ in (a) MS periodic? Justify your answer.

Hint: A process $\boldsymbol{x}(t)$ *is* MS periodic *if, and only if, its autocorrelation function is* doubly periodic, *namely,*

$$R_{xx}(t_1 + mT, t_2 + nT) = R_{xx}(t_1, t_2) \text{ for every integer } m \text{ and } n.$$

(c) Show that $\boldsymbol{x}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n g(t - nT)$ is wide-sense cyclostationary if $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ is zero-mean wide-sense cyclostationary with period M (i.e., $R_{cc}[n, m] = R_{cc}[n + kM, m + kM]$ for every k). What is the period of the wide-sense cyclostationarity of $\boldsymbol{x}(t)$?

Solution.

(a)

$$\mu_x(t) = E[\boldsymbol{x}(t)] = \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n] g(t - nT) = 0$$

and

$$\begin{aligned} R_{xx}(t_1, t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n \boldsymbol{c}_m^*] g(t_1 - nT) g^*(t_2 - mT) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{cc}[n - m] g(t_1 - nT) g^*(t_2 - mT). \end{aligned}$$

Thus, $\mu_x(t + kT) = \mu_x(t) = 0$ and

$$\begin{aligned} & R_{xx}(t_1 + kT, t_2 + kT) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_{cc}[n - m] g(t_1 + kT - nT) g^*(t_2 + kT - mT) \\ & \quad (\text{Let } n' = n - k \text{ and } m' = m - k) \\ &= \sum_{n'=-\infty}^{\infty} \sum_{m'=-\infty}^{\infty} R_{cc}[n' - m'] g(t_1 - n'T) g^*(t_2 - m'T) \\ &= R_{xx}(t_1, t_2). \end{aligned}$$

Therefore, $\boldsymbol{x}(t)$ is WSCS.

(b) $\boldsymbol{x}(t)$ is not necessarily MS periodic (even if it is WSCS). Suppose

$$g(t) = \begin{cases} 1, & 0 \le t < T; \\ 0 & \text{otherwise,} \end{cases}$$

and $R_{cc}[n-m] = \delta[n-m]$, where $\delta[\cdot]$ is the Kronecker delta function. Derive

$$\begin{aligned} R_{xx}(t,t) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{cc}[n-m] g(t-nT) g^*(t-mT) \\ &= \sum_{m=-\infty}^{\infty} |g(t-mT)|^2 = \infty \end{aligned}$$

and

$$\begin{aligned} R_{xx}(t+T,t) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{cc}[n-m] g(t+T-nT) g^*(t-mT) \\ &= \sum_{m=-\infty}^{\infty} g(t+T-mT) g^*(t-mT) = 0. \end{aligned}$$

Thus, $\boldsymbol{x}(t)$ is not MS periodic because $R_{xx}(t+T,t) \neq R_{xx}(t,t)$.

(c)

$$\begin{aligned} R_{xx}(t_1,t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n \boldsymbol{c}_m^*] g(t_1-nT) g^*(t_2-mT) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{cc}[n,m] g(t_1-nT) g^*(t_2-mT) \end{aligned}$$

implies

$$\begin{aligned} & R_{xx}(t_1+MT, t_2+MT) \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{cc}[n,m] g(t_1+MT-nT) g^*(t_2+MT-mT) \\ &\quad (\text{Let } n' = n-M \text{ and } m' = m-M.) \\ &= \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} R_{cc}[n'+M, m'+M] g(t_1-n'T) g^*(t_2-m'T) \\ &= \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} R_{cc}[n',m'] g(t_1-n'T) g^*(t_2-m'T) \\ &= R_{xx}(t_1,t_2). \end{aligned}$$

The period of the wise-sense cyclostationarity of $\boldsymbol{x}(t)$ is MT.

3. (a) If $\boldsymbol{x}(t)$ is BL, then

$$\lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}E\left[|\boldsymbol{x}(t+\tau)-\boldsymbol{x}(t)|^2\right]dt \le \sigma^2\tau^2\bar{R}_{xx}(0),$$

provided the limit exists.

(b) Show that we can improve the upper bound to:

$$\lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}E\left[|\boldsymbol{x}(t+\tau)-\boldsymbol{x}(t)|^2\right]dt \le \min\{\sigma^2\tau^2,4\}\cdot\bar{R}_{xx}(0).$$

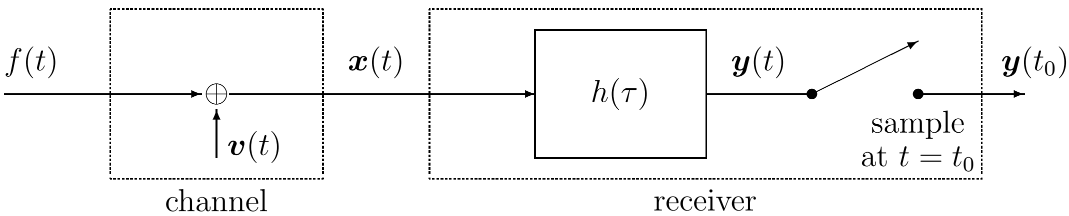
Solution.

(a) See Slide 10-50.

(b) We only need to additionally show that

$$\begin{aligned}\bar{R}_{yy}(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|^2\bar{S}_{xx}(\omega)d\omega\\ &= \frac{1}{2\pi}\int_{-\sigma}^{\sigma}|e^{-j\omega\tau}-1|^2\bar{S}_{xx}(\omega)d\omega\\ &= \frac{1}{2\pi}\int_{-\sigma}^{\sigma}4\sin^2\left(\frac{\omega\tau}{2}\right)\bar{S}_{xx}(\omega)d\omega\\ &\le \frac{1}{2\pi}\int_{-\sigma}^{\sigma}4\bar{S}_{xx}(\omega)d\omega\\ &= 4\cdot\frac{1}{2\pi}\int_{-\sigma}^{\sigma}\bar{S}_{xx}(\omega)d\omega = 4\bar{R}_{xx}(0).\end{aligned}$$

4.



Suppose $h(\tau)$ satisfies that $h(\tau)=0$ for $|\tau|>T$, $h(-\tau)=h(\tau)$, and $\int_{-T}^{T}h(\tau)d\tau=1$. Assume that $\boldsymbol{v}(t)$ is zero-mean white with PSD $S_{vv}(\omega)=\frac{N_0}{2}$.

(a) Subject to $f(t_0-\tau)=a_0+a_1\tau+a_2\tau^2+a_3\tau^3$, where $a_0=f(t_0)$, find the filter $h(\tau)$ that minimizes $E\{[\boldsymbol{y}(t_0)-f(t_0)]^2\}$.

(b) Subject to $f(t_0-\tau)=a_0+a_4\tau^4$, where $a_0=f(t_0)$, find the filter $h(\tau)$ that minimizes $E\{[\boldsymbol{y}(t_0)-f(t_0)]^2\}$.

Solution.

(a) First derive that

$$\begin{aligned}
e &= E\{[\boldsymbol{y}(t_0)-f(t_0)]^2\}\\
&= E\left\{[y_f(t_0)+\boldsymbol{y}_v(t_0)-f(t_0)]^2\right\}\\
&= (y_f(t_0)-f(t_0))^2+E[\boldsymbol{y}_v^2(t_0)] \qquad (\boldsymbol{y}_v(t) \text{ zero mean})\\
&= \left(\int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0)\right)^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)E[\boldsymbol{v}(t_0-u)\boldsymbol{v}(t_0-v)]dudv\\
&= \left(\int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0)\right)^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)\frac{N_0}{2}\delta(v-u)dudv\\
&= \left(\int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0)\right)^2 + \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(v)dv\\
&= b^2+\sigma^2,
\end{aligned}$$

where

$$\text{bias } b=\int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0) \quad \text{and} \quad \text{variance } \sigma^2=\frac{N_0}{2}\int_{-\infty}^{\infty} h^2(v)dv.$$

Observe that

$$\begin{aligned}
b &= \int_{-\infty}^{\infty} h(\tau)\left[a_0+a_1\tau+a_2\tau^2+a_3\tau^3\right]d\tau - f(t_0)\\
&= \left(a_0\int_{-T}^{T} h(\tau)d\tau - f(t_0)\right) + a_1\cancel{\int_{-T}^{T}\tau h(\tau)d\tau} + a_2\int_{-T}^{T}\tau^2 h(\tau)d\tau + a_3\cancel{\int_{-T}^{T}\tau^3 h(\tau)d\tau}\\
&= a_2\int_{-T}^{T}\tau^2 h(\tau)d\tau.
\end{aligned}$$

By Lagrange multiplier technique, we minimize e subject to

$$\int_{-T}^{T} h(\tau)d\tau = 1 \quad \text{and} \quad \int_{-T}^{T}\tau^2 h(\tau)d\tau = c,$$

and obtain:

$$\begin{aligned}
\frac{\partial e}{\partial h(v)} &= \frac{\partial\left[a_2^2c^2+\frac{N_0}{2}\int_{-T}^{T} h^2(\tau)d\tau\right]}{\partial h(v)}\\
&\quad -\frac{\partial\left[\lambda_1\left(\int_{-T}^{T} h(\tau)d\tau - 1\right)+\lambda_2\left(\int_{-T}^{T}\tau^2 h(\tau)d\tau - c\right)\right]}{\partial h(v)}\\
&= N_0 h(v)-\lambda_1-\lambda_2 v^2 = 0.
\end{aligned}$$

This implies

$$h_{\rm opt}(v) = \frac{1}{N_0}\left(\lambda_1 + \lambda_2 v^2\right) \;\text{ for } |v| \le T.$$

Solving

$$\int_{-T}^{T} h_{\rm opt}(\tau)d\tau = \frac{2T}{N_0}\lambda_1 + \frac{2T^3}{3N_0}\lambda_2 = 1 \quad\text{and}\quad \int_{-T}^{T}\tau^2 h_{\rm opt}(\tau)d\tau = \frac{2T^3}{3N_0}\lambda_1 + \frac{2T^5}{5N_0}\lambda_2 = c$$

yields

$$\lambda_1 = -\frac{15N_0}{8T^3}\left(c - \frac{3}{5}T^2\right) \quad\text{and}\quad \lambda_2 = \frac{45N_0}{8T^5}\left(c - \frac{1}{3}T^2\right),$$

and

$$h_{\rm opt}(v) = \frac{15}{8T}\left[\left(3\frac{c}{T^2} - 1\right)\frac{v^2}{T^2} - \left(\frac{c}{T^2} - \frac{3}{5}\right)\right] = \frac{15}{8T}\left[(3\bar{c} - 1)\frac{v^2}{T^2} - \left(\bar{c} - \frac{3}{5}\right)\right]$$

where $\bar{c} = c/T^2$. We continue the derivation:

$$\begin{aligned} e &= a_2^2 c^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau)d\tau \\ &= (a_2 T^2 \bar{c})^2 + \frac{N_0 T}{2}\int_{-1}^{1} h^2(T\bar{\tau})d\bar{\tau} \\ &= (a_2 T^2 \bar{c})^2 + \frac{225N_0}{128T}\int_{-1}^{1}[(3\bar{c}-1)\bar{\tau}^2 - (\bar{c} - 3/5)]^2 d\bar{\tau} \\ &= (a_2 T^2 \bar{c})^2 + \frac{3N_0}{16T}(3 - 10\bar{c} + 15\bar{c}^2). \end{aligned}$$

Consequently,

$$\frac{\partial e}{\partial \bar{c}} = 2a_2^2 T^4 \bar{c} + \frac{15N_0}{8T}(-1 + 3\bar{c}) = 0$$

implies

$$\bar{c}_{\rm min} = \frac{\frac{15N_0}{8T}}{2a_2^2T^4 + \frac{45N_0}{8T}} = \frac{15N_0}{16a_2^2T^5 + 45N_0} = \frac{15}{A + 45},$$

and

$$e_{\rm min} = \frac{N_0}{16T}A\bar{c}_{\rm min}^2 + \frac{3N_0}{16T}(3 - 10\bar{c}_{\rm min} + 15\bar{c}_{\rm min}^2) = \frac{9N_0}{16T}\frac{(A+20)}{(A+45)},$$

where $A = \frac{16a_2^2T^5}{N_0}$. Finally,

$$\begin{aligned} h_{\rm opt}(v) &= \frac{15}{8T}\left[(3\bar{c}_{\rm min} - 1)\frac{v^2}{T^2} - \left(\bar{c}_{\rm min} - \frac{3}{5}\right)\right] \\ &= \frac{15}{8T}\left[-\frac{A}{(A+45)}\frac{v^2}{T^2} + \frac{3(A+20)}{5(A+45)}\right] \\ &= \frac{15A}{8T(A+45)}\left(-\frac{v^2}{T^2} + \frac{3}{5} + \frac{12}{A}\right). \end{aligned}$$

(b) Again, we first derive that

$$e = b^2 + \sigma^2,$$

where

$$\text{bias } b = \int_{-\infty}^{\infty} h(\tau) f(t_0 - \tau) d\tau - f(t_0) \quad \text{and} \quad \text{variance } \sigma^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} h^2(v) dv.$$

Observe that

$$\begin{aligned} b &= \int_{-\infty}^{\infty} h(\tau) \left[a_0 + a_4 \tau^4\right] d\tau - f(t_0) \\ &= \left(a_0 \int_{-T}^{T} h(\tau) d\tau - f(t_0) \right) + a_4 \int_{-T}^{T} \tau^4 h(\tau) d\tau \\ &= a_4 \int_{-T}^{T} \tau^4 h(\tau) d\tau. \end{aligned}$$

By Lagrange multiplier technique, we minimize e subject to

$$\int_{-T}^{T} h(\tau) d\tau = 1 \quad \text{and} \quad \int_{-T}^{T} \tau^4 h(\tau) d\tau = d$$

and obtain:

$$\begin{aligned} \frac{\partial e}{\partial h(v)} &= \frac{\partial \left[(a_4 d)^2 + \frac{N_0}{2} \int_{-T}^{T} h^2(\tau) d\tau \right]}{\partial h(v)} \\ &\quad - \frac{\partial \left[\lambda_1 \left(\int_{-T}^{T} h(\tau) d\tau - 1 \right) + \lambda_3 \left(\int_{-T}^{T} \tau^4 h(\tau) d\tau - d \right) \right]}{\partial h(v)} \\ &= N_0 h(v) - \lambda_1 - \lambda_3 v^4 = 0. \end{aligned}$$

This implies

$$h_{\text{opt}}(v) = \frac{1}{N_0} \left(\lambda_1 + \lambda_3 v^4 \right) \text{ for } |v| \le T.$$

Solving

$$\int_{-T}^{T} h_{\text{opt}}(\tau) d\tau = \frac{2T}{N_0} \lambda_1 + \frac{2T^5}{5N_0} \lambda_3 = 1 \quad \text{and} \quad \int_{-T}^{T} \tau^4 h_{\text{opt}}(\tau) d\tau = \frac{2T^5}{5N_0} \lambda_1 + \frac{2T^9}{9N_0} \lambda_3 = d$$

yields

$$\lambda_1 = \frac{5N_0}{32T^5}(-9d + 5T^4) \quad \text{and} \quad \lambda_3 = \frac{45N_0}{32T^9}(5d - T^4)$$

and

$$h_{\rm opt}(v) = \frac{1}{N_0}\left(\lambda_1 + \lambda_3 v^4\right) = \frac{5}{32T}(-9\bar{d}+5) + \frac{45}{32T}(5\bar{d}-1)\frac{v^4}{T^4}$$

where $\bar{d} = d/T^4$. We continue the derivation:

$$\begin{aligned} e &= (a_4 d)^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau)d\tau \\ &= (a_4T^4\bar{d})^2 + \frac{N_0T}{2}\int_{-1}^{1} h^2(T\bar{\tau})d\bar{\tau} \\ &= (a_4T^4\bar{d})^2 + \frac{25N_0}{2048T}\int_{-1}^{1}[(-9\bar{d}+5)+9(5\bar{d}-1)\bar{\tau}^4]^2 d\bar{\tau} \\ &= (a_4T^4\bar{d})^2 + \frac{5N_0}{64T}(5-18\bar{d}+45\bar{d}^2). \end{aligned}$$

Consequently,

$$\frac{\partial e}{\partial \bar{d}} = 2a_4^2T^8\bar{d} + \frac{45N_0}{32T}(-1+5\bar{d}) = 0$$

implies

$$\bar{d}_{\rm min} = \frac{\frac{45N_0}{32T}}{2a_4^2T^8 + \frac{225N_0}{32T}} = \frac{45N_0}{64a_4^2T^9 + 225N_0} = \frac{45}{A+225},$$

and

$$\begin{aligned} e_{\rm min} &= \frac{N_0}{64T}A\bar{d}_{\rm min}^2 + \frac{5N_0}{64T}(5-18\bar{d}_{\rm min}+45\bar{d}_{\rm min}^2) \\ &= \frac{N_0}{64T}\left(A\bar{d}_{\rm min}^2 + 25 - 90\bar{d}_{\rm min} + 225\bar{d}_{\rm min}^2\right) = \frac{25N_0}{64T}\frac{(A+144)}{(A+225)}, \end{aligned}$$

where $A = \frac{64a_4^2T^9}{N_0}$. Finally,

$$\begin{aligned} h_{\rm opt}(v) &= \frac{5}{32T}(-9\bar{d}_{\rm min}+5) + \frac{45}{32T}(5\bar{d}_{\rm min}-1)\frac{v^4}{T^4} \\ &= \frac{45A}{32T(A+225)}\left(-\frac{v^4}{T^4} + \frac{5}{9} + \frac{80}{A}\right). \end{aligned}$$