2017 Second Quiz for Random Processes

Name ______________ **Student ID** ______________

1. (50%) Show that $\boldsymbol{x}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n g(t - nT)$ is wide-sense cyclostationary (WSCS) if $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ is zero-mean WSCS (i.e., $R_{cc}[n, m] = R_{cc}[n + kM, m + kM]$ for every k).

   Solution. First, $\mu_x(t) = E[\boldsymbol{x}(t)] = \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n]g(t - nT) = 0$ and hence is periodic. Second,

$$
\begin{aligned}
R_{xx}(t_1, t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n \boldsymbol{c}_m^*] g(t_1 - nT) g^*(t_2 - mT) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{cc}[n, m] g(t_1 - nT) g^*(t_2 - mT)
\end{aligned}
$$

   implies

$$
\begin{aligned}
& R_{xx}(t_1 + MT, t_2 + MT) \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R_{cc}[n, m] g(t_1 + MT - nT) g^*(t_2 + MT - mT) \\
&\quad (\text{Let } n' = n - M \text{ and } m' = m - M.) \\
&= \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} R_{cc}[n' + M, m' + M] g(t_1 - n'T) g^*(t_2 - m'T) \\
&= \sum_{m'=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} R_{cc}[n', m'] g(t_1 - n'T) g^*(t_2 - m'T) \\
&= R_{xx}(t_1, t_2).
\end{aligned}
$$

   Therefore, $\boldsymbol{x}(t)$ is WSCS.

2. (50%) Show that for WSS $\boldsymbol{x}(t)$,

$$
\lim_{w \to \infty} \frac{1}{2w} \int_{-w}^{w} E\left[|\boldsymbol{x}(t + \tau) - \boldsymbol{x}(t)|^2 \right] dt \le 4 \cdot R_{xx}(0).
$$

   Hint: Let $\boldsymbol{y}(t)$ *be the output due to input* $\boldsymbol{x}(t)$ *and filter* $H(\omega) = e^{j\omega\tau} - 1$*. Then,* $\boldsymbol{y}(t) = \boldsymbol{x}(t + \tau) - \boldsymbol{x}(t)$*.*

Solution.

$$\begin{aligned}
R_{yy}(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_{xx}(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |e^{j\omega\tau}-1|^2 S_{xx}(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\sin^2\left(\frac{\omega\tau}{2}\right) S_{xx}(\omega)d\omega \\
&\le \frac{1}{2\pi}\int_{-\infty}^{\infty} 4S_{xx}(\omega)d\omega \\
&= 4\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega = 4\cdot R_{xx}(0).
\end{aligned}$$