Sample problems for the 3rd quiz

1. Let E be the (one-sided) event containing

$$(1,0,0,1,0,0,\ldots), \quad (0,0,1,0,0,1\ldots) \quad \text{and} \quad (0,1,0,0,1,0,\ldots),$$

where at each time instance, either 0 or 1 appears.

(a) List the first six components (as did above) of the elements in $\mathbb{T}E$ and $\mathbb{T}^{-1}E$, respectively.

(b) Does the three elements form a $\mathbb{T}$-invariant group?

(c) Is E an ergodic set?

(d) Subject to $\Pr[E] = 1$, what is the ensemble average of the first component?

(e) Subject to $\Pr[E] = 1$, what is the time average?

Solution.

(a)

$$\mathbb{T}E = \{(0,0,1,0,0,1,\ldots), \quad (0,1,0,0,1,0\ldots), \quad (1,0,0,1,0,0,\ldots)\} = E,$$

and

$$\begin{aligned}\mathbb{T}^{-1}E = \{&(0,0,0,1,0,0,\ldots),(1,0,0,1,0,0,\ldots),(0,0,1,0,0,1\ldots),\\ &(1,0,1,0,0,1\ldots),(0,1,0,0,1,0,\ldots),(1,1,0,0,1,0,\ldots)\},\end{aligned}$$

(b) Yes.

(c) No, since $\mathbb{T}^{-1}E \neq E$.

(d) Indeterminate, since we do not know the probability of each of the three elements.

(e) $\frac{1}{3}$

2. Prove the lemma in Slide 12-19, i.e., a WSS process $\boldsymbol{x}(t)$ is mean-ergodic if, and only if, $\lim_{T\to\infty} \mathrm{Var}[\boldsymbol{\eta}_T] = 0$.

Solution.

if-part: For a WSS $\boldsymbol{x}(t)$ with $\eta_x = E[\boldsymbol{x}(t)]$ is a constant (independent of t), we have $E[\boldsymbol{\eta}_T] = \eta_x$ (also independent of t). Hence, if $\lim_{T\to\infty} \mathrm{Var}[\boldsymbol{\eta}_T] = 0$, then there exists $\eta = \eta_x$ such that

$$\begin{aligned}\lim_{T\to\infty} E\left[\,|\boldsymbol{\eta}_T - \eta|^2\,\right] &= \lim_{T\to\infty} E\left[\,|\boldsymbol{\eta}_T - \eta_x|^2\,\right]\\ &= \lim_{T\to\infty} E\left[\,|\boldsymbol{\eta}_T - E[\boldsymbol{\eta}_T]|^2\,\right]\\ &= \lim_{T\to\infty} \mathrm{Var}[\boldsymbol{\eta}_T] = 0.\end{aligned}$$

Thus, $\boldsymbol{x}(t)$ is mean-ergodic.

only-if-part: If there exists a constant η such that $\lim_{T\to\infty} E\left[|\boldsymbol{\eta}_T - \eta|^2\right] = 0$ but $\lim_{T\to\infty} \text{Var}[\boldsymbol{\eta}_T] > 0$, then

$$\begin{aligned}
\lim_{T\to\infty} E\left[|\boldsymbol{\eta}_T - \eta|^2\right] &= \lim_{T\to\infty} E\left[|\boldsymbol{\eta}_T - \eta_x + \eta_x - \eta|^2\right] \\
&= \lim_{T\to\infty} \Big(E\left[|\boldsymbol{\eta}_T - \eta_x|^2\right] + E\left[(\boldsymbol{\eta}_T - \eta_x)(\eta_x - \eta)^*\right] \\
&\quad + E\left[(\eta_x - \eta)(\boldsymbol{\eta}_T - \eta_x)^*\right] + E\left[|\eta_x - \eta|^2\right] \Big).
\end{aligned}$$

By WSS, we have $E\left[(\boldsymbol{\eta}_T - \eta_x)\right] = 0$. Thus,

$$\lim_{T\to\infty} E\left[|\boldsymbol{\eta}_T - \eta|^2\right] = \lim_{T\to\infty} \left(E\left[|\boldsymbol{\eta}_T - \eta_x|^2\right] + E\left[|\eta_x - \eta|^2\right]\right) > 0$$

no matter whether $\eta = \eta_x$ or not; hence, a desired contradiction is obtained.

3. Show that subject to $C_{xx}(\tau) \geq 0$,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} C(\tau) d\tau = 0 \quad \text{if, and only if,} \quad \lim_{\tau\to\infty} |C(\tau)| = 0.$$

Note: This indicates that when $C_{xx}(\tau) \geq 0$, the sufficient condition in Slide 12-24 becomes a necessary condition. In other words, only when $C_{xx}(\tau)$ alternates between positive and negative values, $\lim_{|\tau|\to\infty} |C_{xx}(\tau)| = 0$ is a sufficient condition but not necessarily a necessary condition.

Solution. The "if-part" has been proved in Slide 11-24; so, we only need to take care of the "only-if-part" here.

If $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} C(\tau) d\tau = 0$ but $\lim_{|\tau|\to\infty} |C(\tau)| = \lim_{|\tau|\to\infty} C(\tau) > \eta$ for some $\eta > 0$, then there exists T_0 such that for $|\tau| > T_0$, $C(\tau) > \eta$. Accordingly,

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} C(\tau) d\tau &\geq \lim_{T\to\infty} \frac{1}{2T} \int_{T_0<|\tau|<T} C(\tau) d\tau \\
&\geq \lim_{T\to\infty} \frac{1}{2T} \int_{T_0<|\tau|<T} \eta d\tau \\
&= \frac{\eta}{2} > 0,
\end{aligned}$$

which contradicts to $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} C(\tau) d\tau = 0$. Therefore,

$$\lim_{|\tau|\to\infty} |C(\tau)| = \lim_{|\tau|\to\infty} C(\tau) = 0.$$

4. Let $\boldsymbol{y}(t) \triangleq \boldsymbol{x}(t)c(t)$, where $\boldsymbol{x}(t)$ is a zero-mean **real-valued** random process, and $c(t)$ is a **real-valued** function of t. Define

$$\boldsymbol{Y}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{y}(t)e^{-j\omega t}dt.$$

(a) Show that if $\boldsymbol{x}(t)$ is WSS and $c(t)$ is symmetric (i.e., $c(t) = c(-t)$), then

$$S_{yy}(u, v) \triangleq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t, s)e^{-j(ut+vs)}dtds$$

is real-valued.

(b) Is $S_{yy}(u, v)$ real-valued for general $\boldsymbol{x}(t)$ (not necessarily WSS) and $c(t)$ (not necessarily symmetric)?

(c) Prove that if $\boldsymbol{x}(t)$ is a Gaussian random process,

$$\text{Cov}\{|\boldsymbol{Y}(u)|^2, |\boldsymbol{Y}(v)|^2\} = |S_{yy}(u, -v)|^2 + |S_{yy}(u, v)|^2.$$

Solution.

(a) By

$$\begin{aligned} R_{yy}(t, s) &= E[\boldsymbol{y}(t)\boldsymbol{y}^*(s)] = E[\boldsymbol{x}(t)c(t)\boldsymbol{x}^*(s)c^*(s)] \\ &= E[\boldsymbol{x}(t)\boldsymbol{x}^*(s)]c(t)c^*(s) = R_{xx}(t, s)c(t)c^*(s), \end{aligned}$$

we derive

$$\begin{aligned} S_{yy}(u, v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t, s)c(t)c^*(s)e^{-j(ut+vs)}dtds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t - s)c(t)c^*(s)e^{-j(ut+vs)}dtds \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(f)e^{j(t-s)f}df\right) c(t)c^*(s)e^{-j(ut+vs)}dtds \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} c(t)e^{-j(u-f)t}dt\right)\left(\int_{-\infty}^{\infty} c^*(s)e^{-j(v+f)s}ds\right) S_{xx}(f)df \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} c(t)e^{-j(u-f)t}dt\right)\left(\int_{-\infty}^{\infty} c(s)e^{-j(-v-f)s}ds\right)^* S_{xx}(f)df \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}C(u - f)C^*(-v - f)S_{xx}(f)df \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}C(u - f)C(v + f)S_{xx}(f)df. \end{aligned}$$

The proof is completed by noting that the Fourier transform $C(\omega)$ of a real symmetric function $c(t)$ is real symmetric.

See Slide 12-46 for the same situation.

(b) For general real-valued $c(t)$ (not necessarily symmetric) and real-valued WSS $\boldsymbol{x}(t)$, we can derive, as similar to (a), that

$$\begin{aligned}
S_{yy}(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t,s)c(t)c^*(s)e^{-j(ut+vs)}dtds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t-s)c(t)c^*(s)e^{-j(ut+vs)}dtds \\
&= \cdots \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} C(u-f)C(v+f)S_{xx}(f)df.
\end{aligned}$$

Since $C(u)$ is no longer guaranteed to be real-valued, $S_{yy}(u,v)$ may be complex-valued. If we further relax $\boldsymbol{x}(t)$ to be possibly non-WSS, $S_{yy}(u,v)$ can be surely made complex-valued.

(c) Let $\boldsymbol{Y}(u) = \boldsymbol{A}(u) + j\boldsymbol{B}(u)$ and $\boldsymbol{Y}(v) = \boldsymbol{A}(v) + j\boldsymbol{B}(v)$, where $\boldsymbol{A}(u)$, $\boldsymbol{B}(u)$, $\boldsymbol{A}(v)$ and $\boldsymbol{B}(v)$ are jointly Gaussian (because $\boldsymbol{y}(t)$ is Gaussian). Derive

$$\begin{aligned}
&\text{Cov}\{|\boldsymbol{Y}(u)|^2, |\boldsymbol{Y}(v)|^2\} \\
&\quad= E[(\boldsymbol{A}^2(u)+\boldsymbol{B}^2(u))(\boldsymbol{A}^2(v)+\boldsymbol{B}^2(v))] - E[\boldsymbol{A}^2(u)+\boldsymbol{B}^2(u)]E[\boldsymbol{A}^2(v)+\boldsymbol{B}^2(v)] \\
&\quad= E[\boldsymbol{A}^2(u)\boldsymbol{A}^2(v)] + E[\boldsymbol{A}^2(u)\boldsymbol{B}^2(v)] + E[\boldsymbol{B}^2(u)\boldsymbol{A}^2(v)] + E[\boldsymbol{B}^2(u)\boldsymbol{B}^2(v)] \\
&\qquad -E[\boldsymbol{A}^2(u)]E[\boldsymbol{A}^2(v)] - E[\boldsymbol{A}^2(u)]E[\boldsymbol{B}^2(v)] - E[\boldsymbol{B}^2(u)]E[\boldsymbol{A}^2(v)] - E[\boldsymbol{B}^2(u)]E[\boldsymbol{B}^2(v)] \\
&\quad= 2\underbrace{E^2[\boldsymbol{A}(u)\boldsymbol{A}(v)]}_{C^2} + 2\underbrace{E^2[\boldsymbol{A}(u)\boldsymbol{B}(v)]}_{E^2} + 2\underbrace{E^2[\boldsymbol{B}(u)\boldsymbol{A}(v)]}_{F^2} + 2\underbrace{E^2[\boldsymbol{B}(u)\boldsymbol{B}(v)]}_{D^2},
\end{aligned}$$

where the last step follow from the fact that

$$E[\boldsymbol{x}^2\boldsymbol{y}^2] - E[\boldsymbol{x}^2]E[\boldsymbol{y}^2] = 2E^2[\boldsymbol{x}\boldsymbol{y}]$$

for any jointly Gaussian $(\boldsymbol{x}, \boldsymbol{y})$. We further derive

$$\begin{aligned}
S_{yy}(u,-v) &= R_{YY}(u,v) = E[\boldsymbol{Y}(u)\boldsymbol{Y}^*(v)] \\
&= E[(\boldsymbol{A}(u)+j\boldsymbol{B}(u))(\boldsymbol{A}(v)-j\boldsymbol{B}(v))] \\
&= \underbrace{E[\boldsymbol{A}(u)\boldsymbol{A}(v)]}_{C} + \underbrace{E[\boldsymbol{B}(u)\boldsymbol{B}(v)]}_{D} - j\underbrace{E[\boldsymbol{A}(u)\boldsymbol{B}(v)]}_{E} + j\underbrace{E[\boldsymbol{B}(u)\boldsymbol{A}(v)]}_{F}
\end{aligned}$$

and

$$
\begin{aligned}
S_{yy}(u,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{yy}(t_1,t_2)e^{-j(ut_1+vt_2)}dt_1dt_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\boldsymbol{y}(t_1)\boldsymbol{y}^*(t_2)]e^{-j(ut_1+vt_2)}dt_1dt_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\boldsymbol{y}(t_1)\boldsymbol{y}(t_2)]e^{-j(ut_1+vt_2)}dt_1dt_2 \\
&= E\left[\left(\int_{-\infty}^{\infty}\boldsymbol{y}(t_1)e^{-jut_1}dt_1\right)\left(\int_{-\infty}^{\infty}\boldsymbol{y}(t_2)e^{-jvt_2}dt_2\right)\right] \\
&= E[\boldsymbol{Y}(u)\boldsymbol{Y}(v)] \\
&= E[(\boldsymbol{A}(u)+j\boldsymbol{B}(u))(\boldsymbol{A}(v)+j\boldsymbol{B}(v))] \\
&= \underbrace{E[\boldsymbol{A}(u)\boldsymbol{A}(v)]}_{C} - \underbrace{E[\boldsymbol{B}(u)\boldsymbol{B}(v)]}_{D} + j\underbrace{E[\boldsymbol{A}(u)\boldsymbol{B}(v)]}_{E} + j\underbrace{E[\boldsymbol{B}(u)\boldsymbol{A}(v)]}_{F}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
2C &= \mathrm{Re}\{S_{yy}(u,-v)\} + \mathrm{Re}\{S_{yy}(u,v)\} \\
2D &= \mathrm{Re}\{S_{yy}(u,-v)\} - \mathrm{Re}\{S_{yy}(u,v)\} \\
2E &= -\mathrm{Im}\{S_{yy}(u,-v)\} + \mathrm{Im}\{S_{yy}(u,v)\} \\
2F &= \mathrm{Im}\{S_{yy}(u,-v)\} + \mathrm{Im}\{S_{yy}(u,v)\}
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
4C^2+4D^2+4E^2+4F^2 &= 2\left(\mathrm{Re}\{S_{yy}(u,-v)\}\right)^2 + 2\left(\mathrm{Re}\{S_{yy}(u,v)\}\right)^2 \\
&\quad +2\left(\mathrm{Im}\{S_{yy}(u,-v)\}\right)^2 + 2\left(\mathrm{Im}\{S_{yy}(u,v)\}\right)^2 \\
&= 2|S_{yy}(u,-v)|^2 + 2|S_{yy}(u,v)|^2.
\end{aligned}
$$