2017 3rd Quiz for Random Processes

Name ______________ **Student ID** ______________

1. Re-define $\mathbb{T}$ operator as the right circular shift of (x_1, x_2, x_3, x_4), where each $x_i \in \{0, 1\}$, i.e.,

$$\mathbb{T}(x_1, x_2, x_3, x_4) = (x_4, x_1, x_2, x_3).$$

Re-define that a set E is an ergodic set if $\mathbb{T}^{-1}E = E$, where $\mathbb{T}^{-1}$ is the left circular shift operator.

(a) (20%) Is $G = \{(1, 0, 0, 1), (0, 0, 1, 1), (0, 1, 1, 0)\}$ an ergodic set (under the newly defined $\mathbb{T}$)? Justify your answer by one sentence.

(b) (30%) List all the indecomposable non-empty ergodic sets for zero-one sequences of length 4. Note that a set is an "indecomposable" ergodic set E if no proper subset of E is an ergodic set.

(c) (30%) List the time average for each of the indecomposable non-empty ergodic set in (b).

(d) (20%) A process $\{X(t), t \in \mathcal{I}\}$, where $\mathcal{I} = \{1, 2, 3, 4\}$, is *ergodic* if any ergodic set has probability either 1 or 0. Give an example of an ergodic process based on (b).

Solution.

(a) Since $\mathbb{T}^{-1}E = \{(0, 0, 1, 1), (0, 1, 1, 0), (1, 1, 0, 0)\} \neq E$, it is not an ergodic set.

(b) There are six of them, which are
$E_1 = \{(0, 0, 0, 0)\}$,
$E_2 = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$,
$E_3 = \{(1, 0, 1, 0), (0, 1, 0, 1)\}$,
$E_4 = \{(0, 0, 1, 1), (0, 1, 1, 0), (1, 1, 0, 0), (1, 0, 0, 1)\}$,
$E_5 = \{(1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 1)\}$,
$E_6 = \{1, 1, 1, 1\}$.

(c) The time averages for E_1, E_2, E_3, E_4, E_5 and E_6 are respectively 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{4}$ and 1.

(d) By definition of an ergodic process, only one of $\Pr(E_1)$, $\Pr(E_2)$, $\Pr(E_3)$, $\Pr(E_4)$, $\Pr(E_5)$ and $\Pr(E_6)$ is equal to 1, and the remaining must be zero. So, let a random process X_1, X_2, X_3, X_4 satisfy $\Pr((X_1, X_2, X_3, X_4) \in E_3) = 1$. Then, it is an ergodic process.

Note: For the ergodic process just defined, we have $\Pr\left[\frac{X_1 + X_2 + X_3 + X_4}{4} = \frac{1}{2}\right] = 1.$