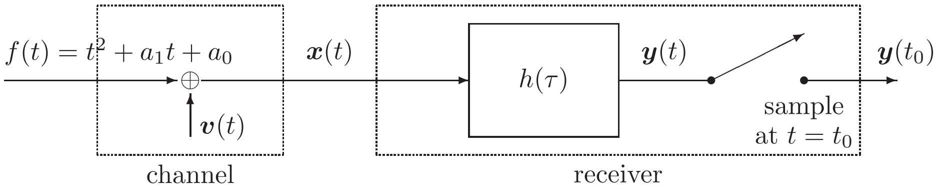
2012 Second Quiz for Random Processes

Name _______________ **Student ID** _______________



1. (50%) For the system above, a center problem in communications is to design the filter $h(\tau)$ such that $e \triangleq E\{[\boldsymbol{y}(t_0) - f(t_0)]^2\}$ is minimized, provided $\boldsymbol{v}(t)$ is zero-mean white (i.e., $R_{vv}(t, t+\tau) = \frac{N_0}{2}\delta(\tau)$). After some basic derivation, we obtain:

$$e = E\{[\boldsymbol{y}(t_0) - f(t_0)]^2\} = b^2 + \sigma^2,$$

where bias $b = \left(\int_{-\infty}^{\infty} h(\tau) f(t_0 - \tau) d\tau\right) - f(t_0)$ and variance $\sigma^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(v) dv$. Show that

$$b \quad = \quad \int_{-1}^{1} \tau^2 h(\tau) d\tau.$$

if $f(t) = t^2 + a_1 t + a_2$, and $h(\tau)$ satisfies (i) $h(\tau) = 0$ for $|\tau| > 1$, (ii) $h(-\tau) = h(\tau)$, and (iii) $\int_{-1}^{1} h(\tau) dt = 1$.

2. (50%) By Lagrange multiplier technique, we minimize e subject to

$$\int_{-1}^{1} h(\tau) d\tau = 1 \quad \text{and} \quad \int_{-1}^{1} \tau^2 h(\tau) d\tau = b,$$

and obtain:

$$\begin{aligned} \frac{\partial e}{\partial h(v)} &= \frac{\partial \left[b^2 + \frac{N_0}{2}\int_{-1}^{1} h^2(\tau) d\tau - \lambda_1 \left(\int_{-1}^{1} h(\tau) d\tau - 1\right) - \lambda_2 \left(\int_{-1}^{1} \tau^2 h(\tau) d\tau - b\right)\right]}{\partial h(v)} \\ &= N_0 h(v) - \lambda_1 - \lambda_2 v^2 = 0. \end{aligned}$$

This implies

$$h_{\text{opt}}(v) = \frac{1}{N_0}\left(\lambda_1 + \lambda_2 v^2\right) \text{ for } |v| \le 1.$$

Suppose $N_0 = 1$ and $b = \frac{3}{5}$ for simplicity. Please solve λ_1 and λ_2 and give the corresponding optimal filter $h_{\text{opt}}(v)$.

Solution.

1. See Slide 10-27.
2. Taking $h_{\text{opt}}(v) = \lambda_1 + \lambda_2 v^2$ for $|v| \le 1$ into $\int_{-1}^{1} h(\tau) d\tau = 1$ and $\int_{-1}^{1} \tau^2 h(\tau) d\tau = \frac{3}{5}$ yields

$$\int_{-1}^{1} h_{\text{opt}}(\tau) d\tau = 2\lambda_1 + \frac{2}{3}\lambda_2 = 1 \quad \text{and} \quad \int_{-1}^{1} \tau^2 h_{\text{opt}}(\tau) d\tau = \frac{2}{3}\lambda_1 + \frac{2}{5}\lambda_2 = \frac{3}{5},$$

which implies

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = \frac{2}{3}, \quad \text{and} \quad h_{\text{opt}}(v) = \frac{3}{2}v^2.$$