2007 Third Quiz for Random Processes

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Suppose that $\{\varphi_n(t)\}_{n=1}^{\infty}$ are orthonormal functions in the sense that

$$\int_0^T \varphi_n(t)\varphi_m^*(t)dt = \delta[n-m] \quad \text{for every } n, m \geq 1,$$

where $\delta[\cdot]$ is the Kronecker function. Let $\boldsymbol{c}_n \triangleq \int_0^T \boldsymbol{x}(t)\varphi_n^*(t)dt$, where $\boldsymbol{x}(t)$ is a WSS random process. If the autocorrelation function $R_{xx}(\tau) = E[\boldsymbol{x}(t+\tau)\boldsymbol{x}^*(t)]$ of random process $\boldsymbol{x}(t)$ satisfies

$$\int_0^T R_{xx}(t-s)\varphi_n(s)ds = \varphi_n(t) \quad \text{for every } n \geq 1,$$

prove that $\{\boldsymbol{c}_n\}_{n=1}^{\infty}$ are orthonormal in the sense that $E[\boldsymbol{c}_n\boldsymbol{c}_m^*] = \delta[n-m]$.

Solutions.

$$\begin{aligned}
E[\boldsymbol{c}_n\boldsymbol{c}_m^*] &= E\left[\left(\int_0^T \boldsymbol{x}(t)\varphi_n^*(t)dt\right)\left(\int_0^T \boldsymbol{x}(s)\varphi_m^*(s)ds\right)^*\right] \\
&= \int_0^T\int_0^T E\left[\boldsymbol{x}(t)\boldsymbol{x}^*(s)\right]\varphi_n^*(t)\varphi_m(s)dtds \\
&= \int_0^T\left(\int_0^T R_{xx}(t-s)\varphi_m(s)ds\right)\varphi_n^*(t)dt \\
&= \int_0^T \varphi_m(t)\varphi_n^*(t)dt \\
&= \delta[m-n].
\end{aligned}$$