2012 First Midterm for Random Processes

1. (a) (8%) Find the autocorrelaton function of $\boldsymbol{x}(t)$, where

$$\boldsymbol{x}(t) = \begin{cases} 1, & \text{if } \boldsymbol{n}(0,t) \text{ is even;} \\ -1, & \text{if } \boldsymbol{n}(0,t) \text{ is odd,} \end{cases}$$

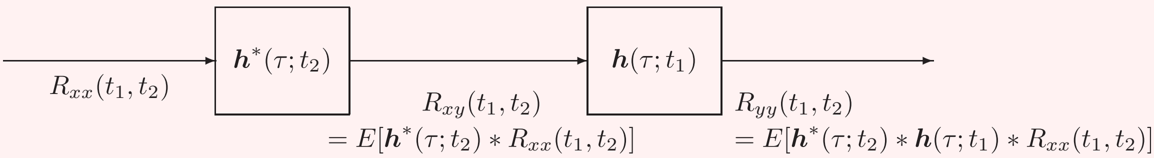
and the process $\boldsymbol{n}(t_1, t_2)$ satisfies

- $\Pr\{\boldsymbol{n}(t_1,t_2) = k\} = \left(\frac{t_2-t_1}{t_2-t_1+1}\right)^k \left(1 - \frac{t_2-t_1}{t_2-t_1+1}\right)$ for $k = 0, 1, 2, \ldots$ and $t_2 \geq t_1$,
- and $\boldsymbol{n}(t_1, t_2)$ and $\boldsymbol{n}(t_3, t_4)$ are independent if (t_1, t_2) and (t_3, t_4) are non-overlapping intervals.

(b) (6%) Is process $\boldsymbol{x}(t)$ WSS? Justify your answer.

(c) (8%) Pass $\boldsymbol{x}(t)$ through a differentiator. Find the autocorrelation function of the output process $\boldsymbol{y}(t)$. (Hint: Recall that for the computation of the input-output cross-correlation function $R_{xy}(t_1, t_2)$, Theorem 9-2 below can be reduced to $R_{xy}(t_1, t_2) = \partial R_{xx}(t_1, t_2)/\partial t_2$ if the system is a differentiator.)

Theorem 9-2 For any linear system,



Solution. For notational convenience, I denote $p = \frac{t_2-t_1}{t_2-t_1+1}$ in the following derivation.

(a) For $t_1 \leq t_2$,

$$\begin{aligned}
&E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] \\
&= \Pr[\boldsymbol{n}(0,t_1) = \text{even} \wedge \boldsymbol{n}(0,t_2) = \text{even}] + \Pr[\boldsymbol{n}(0,t_1) = \text{odd} \wedge \boldsymbol{n}(0,t_2) = \text{odd}] \\
&- \Pr[\boldsymbol{n}(0,t_1) = \text{even} \wedge \boldsymbol{n}(0,t_2) = \text{odd}] - \Pr[\boldsymbol{n}(0,t_1) = \text{odd} \wedge \boldsymbol{n}(0,t_2) = \text{even}] \\
&= \Pr[\boldsymbol{n}(0,t_1) = \text{even} \wedge \boldsymbol{n}(t_1,t_2) = \text{even}] + \Pr[\boldsymbol{n}(0,t_1) = \text{odd} \wedge \boldsymbol{n}(t_1,t_2) = \text{even}] \\
&- \Pr[\boldsymbol{n}(0,t_1) = \text{even} \wedge \boldsymbol{n}(t_1,t_2) = \text{odd}] - \Pr[\boldsymbol{n}(0,t_1) = \text{odd} \wedge \boldsymbol{n}(t_1,t_2) = \text{odd}] \\
&= \Pr[\boldsymbol{n}(0,t_1) = \text{even}]\Pr[\boldsymbol{n}(t_1,t_2) = \text{even}] + \Pr[\boldsymbol{n}(0,t_1) = \text{odd}]\Pr[\boldsymbol{n}(t_1,t_2) = \text{even}] \\
&- \Pr[\boldsymbol{n}(0,t_1) = \text{even}]\Pr[\boldsymbol{n}(t_1,t_2) = \text{odd}] - \Pr[\boldsymbol{n}(0,t_1) = \text{odd}]\Pr[\boldsymbol{n}(t_1,t_2) = \text{odd}] \\
&= (\Pr[\boldsymbol{n}(0,t_1) = \text{even} + \Pr[\boldsymbol{n}(0,t_1) = \text{odd}])(\Pr[\boldsymbol{n}(t_1,t_2) = \text{even}] - \Pr[\boldsymbol{n}(t_1,t_2) = \text{odd}]) \\
&= \Pr[\boldsymbol{n}(t_1,t_2) = \text{even}] - \Pr[\boldsymbol{n}(t_1,t_2) = \text{odd}] \\
&= (1-p)(p^0 + p^2 + p^4 + \cdots) - (1-p)(p^1 + p^3 + p^5 + \cdots) \\
&= (1-p)\frac{1}{(1-p^2)} - (1-p)\frac{p}{(1-p^2)} \\
&= \frac{1-p}{1+p} \\
&= \frac{1}{2(t_2-t_1)+1}
\end{aligned}$$

Similarly, for $t_1 > t_2$,

$$E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] = \frac{1}{2(t_1 - t_2) + 1}.$$

Therefore,

$$R_{xx}(t_1, t_2) = E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] = \frac{1}{2|t_1 - t_2| + 1}.$$

(b)

$$\begin{aligned}
E[\boldsymbol{x}(t)] &= 1 \cdot \Pr[\boldsymbol{n}(0,t) = 0, 2, 4, \cdots] + (-1) \cdot \Pr[\boldsymbol{n}(0,t) = 1, 3, 5, \cdots] \\
&= 1 \cdot (1-p)\left[p^0 + p^2 + p^4 + \cdots\right] + (-1) \cdot (1-p)\left[p^1 + p^3 + p^5 + \cdots\right] \\
&= 1 \cdot (1-p)\frac{1}{(1-p^2)} + (-1) \cdot (1-p)\frac{p}{(1-p^2)} \\
&= \frac{1-p}{1+p} \\
&= \frac{1}{2t+1}
\end{aligned}$$

Since the mean function is a function of time t, $\boldsymbol{x}(t)$ is not WSS!

(c)

$$\begin{aligned}
R_{yy}(t_1, t_2) &= \frac{\partial^2 R_{xx}(t_1, t_2)}{\partial t_1 \partial t_2} \\
&= \frac{\partial^2 \left(\frac{1}{2|t_1 - t_2| + 1}\right)}{\partial t_1 \partial t_2} \\
&= \begin{cases} \frac{\partial^2 \left(\frac{1}{2(t_2 - t_1) + 1}\right)}{\partial t_1 \partial t_2} & t_2 > t_1 \\ \frac{\partial^2 \left(\frac{1}{2(t_1 - t_2) + 1}\right)}{\partial t_1 \partial t_2} & t_2 < t_1 \end{cases} \\
&= -\frac{8}{(2(t_2 - t_1) + 1)^3} \text{ for } t_2 \neq t_1
\end{aligned}$$

2. (a) (6%) Given an arbitrary non-negative integrable function $S(\omega)$, construct a complex WSS process $\boldsymbol{x}(t)$ whose power spectrum is equal to $S(\omega)$. You shall justify your construction. (Hint: $e^{j(\boldsymbol{\omega} t - \boldsymbol{\varphi})}$)

   (b) (6%) If $S(\omega)$ is an odd function, is it possible to construct a real WSS process whose power spectrum is equal to $S(\omega)$? Justify your answer.

   (c) (6%) If $S(\omega) = \frac{4\sin(\omega)}{\omega}$, is it possible to construct a WSS process whose power spectrum is equal to $S(\omega)$? Justify your answer. (Hint: $\int_{-1}^{1} 2e^{-j\omega\tau} d\tau = \frac{4\sin(\omega)}{\omega}$.)

Solution.

(a) See Slide 9-95.

(b) The power spectrum density of a real WSS process is always an even function. So the answer is negative.

(c) See Slide 9-109 with $a = 1$. So by the corollary on slide 9-108, the answer is negative.

3. (12%) Suppose that $\{\boldsymbol{c}_i\}$ are uncorrelated with zero mean. Show that the process $\boldsymbol{x}[t] = \sum_i \boldsymbol{c}_i e^{j\omega_i t}$ is WSS.

Solution. See Slide 9-128.

4. Suppose $\boldsymbol{Y}(\omega) = \boldsymbol{X}(\omega)H(\omega)$, where $\boldsymbol{X}(\omega)$ and $\boldsymbol{Y}(\omega)$ are respectively the Fourier transforms of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$. Assume that $\boldsymbol{x}(t)$ is WSS. Prove that if $R_{xx}(\tau) = R_{yy}(\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$, then

   (a) (6%) $|H(\omega)|^2 = 1$;

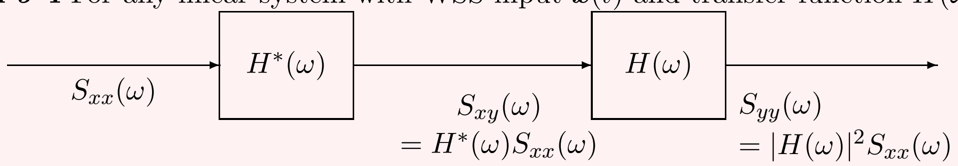
   (b) (6%) $H(\omega) = -H^*(\omega)$;

   (c) (6%) $H(-\omega) = -H(\omega)$.

   (d) (6%) Give a $H(\omega)$ that satisfies (a), (b) and (c).

Hint: $\boldsymbol{X}(\omega) = [1/H(\omega)]\boldsymbol{Y}(\omega)$, $S_{xy}(-\omega) = -S_{xy}(\omega)$ and $S_{xx}(\omega) = S_{xx}(-\omega)$.

Theorem 9-4 For any linear system with WSS input $\boldsymbol{x}(t)$ and transfer function $H(\omega)$,



Solution. See Slides 10-9 and 10-10.

5. Suppose that $\boldsymbol{x}(t)$ is a zero-mean WSCS process with period T (but is not WSS), and suppose that $\boldsymbol{y}(t) = \boldsymbol{x}(t - \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is independent of $\boldsymbol{x}(t)$.

   (a) (8%) Construct a distribution of $\boldsymbol{\theta}$ such that $\boldsymbol{y}(t)$ is WSS. You shall justify your construction.

   (b) (6%) Construct a distribution of $\boldsymbol{\theta}$ such that $\boldsymbol{y}(t)$ is not WSS. You shall justify your construction.

Solution.

(a) $\boldsymbol{\theta}$ is uniformly distributed over $(0, T]$. The justification follows Slide 10-33.

(b) $\Pr[\boldsymbol{\theta} = 0] = 1$. Then $\boldsymbol{y}(t) = \boldsymbol{x}(t)$. So, $\boldsymbol{y}(t)$ is WSS if, and only if, $\boldsymbol{x}(t)$ is WSS.

6. (10%) Suppose that

$$e^{j\omega\tau} = \sum_{n=-\infty}^{\infty} a_{n,\tau} e^{jnT\omega}.$$

Prove that

$$\boldsymbol{x}(t + \tau) = \sum_{n=-\infty}^{\infty} a_{n,\tau} \boldsymbol{x}(t + nT).$$

Solution. Passing the process $\boldsymbol{x}(t)$ via filter $H_1(\omega) = e^{j\omega\tau}$ and filter $H_2(\omega) = \sum_{n=-\infty}^{\infty} a_{n,\tau} e^{jnT\omega}$ will result in the same output. Accordingly,

$$\boldsymbol{x}(t+\tau) = \sum_{n=-\infty}^{\infty} a_{n,\tau} \boldsymbol{x}(t+nT).$$