2012 Second Midterm for Random Processes

1. Suppose the power spectrum of a (discrete) regular WSS process $\boldsymbol{x}[t]$ is

$$S_{xx}[\omega] = \frac{5 - 4\cos(\omega)}{10 - 6\cos(\omega)}.$$

(a) (10%) Find its innovation filter $\mathtt{L}[z]$ that is analytic for $|z| > 1$.
Hint: (i) $\cos(\omega) = (z + z^{-1})/2$, and (ii) $S_{xx}[z] = \mathtt{L}[z]\mathtt{L}[1/z]$ and (iii) roots inside the unit circle.

(b) (10%) Find the impulse response $\mathtt{l}[\tau]$ of the innovation filter $\mathtt{L}[z]$ in (a).
Hint: $\mathtt{l}[\tau] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathtt{L}[\omega] e^{j\omega\tau} d\omega$

(c) (10%) Determine $S_{xx}^{+}[z]$ if we rewrite $S_{xx}[z]$ as

$$S_{xx}[z] = S_{xx}^{+}[z] + S_{xx}^{+}[1/z] - \sum_{i=1}^{n} \alpha_i,$$

where

$$R_{xx}^{+}[\tau] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}^{+}[e^{j\omega}] e^{j\omega\tau} d\omega = 0 \text{ for } \tau < 0,$$

and $\alpha_k = \gamma_k \mathtt{L}[z_k^{-1}]$.
Hint: Start from $\mathtt{L}[z] = \frac{\gamma_i}{1 - z_i z^{-1}}$ and observe that

$$\begin{aligned} S_{xx}[z] &= \mathtt{L}[z]\mathtt{L}[z^{-1}] = \left(\sum_{i=1}^{n} \frac{\gamma_i}{1 - z_i z^{-1}}\right)\left(\sum_{k=1}^{n} \frac{\gamma_k}{1 - z_k z}\right) \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\gamma_i \gamma_k}{(1 - z_i z^{-1})(1 - z_k z)} \\ &= \sum_{i=1}^{n}\sum_{k=1}^{n} \left(\frac{-}{1 - z_i z^{-1}} + \frac{-}{1 - z_k z} - \cdots\right) \end{aligned}$$

Definition (discrete regular processes) A (discrete) process $\boldsymbol{x}[t]$ is regular if there exists a $\mathtt{L}[z]$ such that

$$S_{xx}[\omega] = |\mathtt{L}[e^{j\omega}]|^2$$

where $\mathtt{L}[z]$ $(z = e^{j\omega})$ is analytic for $|z| > 1$.

Solution.

(a) $S_{xx}[z] = \frac{5 - 2(z + z^{-1})}{10 - 3(z + z^{-1})} = \frac{2(z - 1/2)(z - 2)}{3(z - 1/3)(z - 3)} = \frac{2(z - 1/2)}{3(z - 1/3)} \cdot \frac{2(z^{-1} - 1/2)}{3(z^{-1} - 1/3)}$

$\Rightarrow \mathtt{L}[z] = \frac{2(z - 1/2)}{3(z - 1/3)}$

(b) From (a), $\mathtt{L}[\omega] = \frac{2(e^{j\omega} - 1/2)}{3(e^{j\omega} - 1/3)}$. Then,

$$\begin{aligned}
\mathtt{l}[\tau] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathtt{L}[\omega] e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2(e^{j\omega} - 1/2)}{3(e^{j\omega} - 1/3)} e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - \frac{e^{j\omega}}{3e^{j\omega} - 1}\right) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - \frac{1/3}{1 - (1/3)e^{-j\omega}}\right) e^{j\omega\tau} d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(1 - 3^{-1}\left[1 + 3^{-1}e^{-j\omega} + 3^{-2}e^{-j2\omega} + \cdots\right]\right) e^{j\omega\tau} d\omega \\
&= \begin{cases} 0, & \tau < 0 \\ 1 - 3^{-1}, & \tau = 0 \\ -3^{-(1+\tau)}, & \tau > 0 \end{cases}
\end{aligned}$$

(c) $\mathtt{L}[z]$ can be expanded into partial fractions as:

$$\mathtt{L}[z] = \sum_{i=1}^{n} \frac{\gamma_i}{1 - z_i z^{-1}} = \frac{2[1 - (1/2)z^{-1}]}{3[1 - (1/3)z^{-1}]} = \frac{\gamma_1}{1 - z_1 z^{-1}} + \frac{\gamma_2}{1 - z_2 z^{-1}}.$$

where $z_1 = 0$, $z_2 = 1/3$, $\gamma_1 = 1$ and $\gamma_2 = -\frac{1}{3}$. We can then derive:

$$\begin{aligned}
&S_{xx}[z] \\
&= \mathtt{L}[z]\mathtt{L}[z^{-1}] \\
&= \left(1 + \frac{(-1/3)}{1 - (1/3)z^{-1}}\right)\left(1 + \frac{(-1/3)}{1 - (1/3)z}\right) \\
&= 1 \times 1 + 1 \times \frac{(-1/3)}{1 - (1/3)z} + \frac{(-1/3)}{1 - (1/3)z^{-1}} \times 1 + \frac{(-1/3)}{1 - (1/3)z^{-1}} \times \frac{(-1/3)}{1 - (1/3)z} \\
&= (1 \times 1 + 1 \times 1 - 1 \times 1) + \left(1 \times \left(-\frac{1}{3}\right) + 1 \times \frac{(-1/3)}{1 - (1/3)z} - 1 \times \left(-\frac{1}{3}\right)\right) \\
&+ \left(\frac{(-1/3)}{1 - (1/3)z^{-1}} \times 1 + \left(-\frac{1}{3}\right) \times 1 - \left(-\frac{1}{3}\right) \times 1\right) \\
&+ \left(\frac{(-1/3)(-1/3)/[1 - (1/3)(1/3)]}{1 - (1/3)z^{-1}} + \frac{(-1/3)(-1/3)/[1 - (1/3)(1/3)]}{1 - (1/3)z} - \frac{(-1/3)(-1/3)}{1 - (1/3)(1/3)}\right) \\
&= \left(1 \times 1 + 1 \times \left(-\frac{1}{3}\right) + \frac{(-1/3)}{1 - (1/3)z^{-1}} \times 1 + \frac{(-1/3)(-1/3)/[1 - (1/3)(1/3)]}{1 - (1/3)z^{-1}}\right) \\
&+ \left(1 \times 1 + 1 \times \frac{(-1/3)}{1 - (1/3)z} + \left(-\frac{1}{3}\right) \times 1 + \frac{(-1/3)(-1/3)/[1 - (1/3)(1/3)]}{1 - (1/3)z}\right) \\
&- \left(1 \times 1 + 1 \times \left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right) \times 1 + \frac{(-1/3)(-1/3)}{1 - (1/3)(1/3)}\right) \\
&= \left(\frac{2}{3} + \frac{(-5/24)}{1 - (1/3)z^{-1}}\right) + \left(\frac{2}{3} + \frac{(-5/24)}{1 - (1/3)z}\right) - \frac{11}{24}
\end{aligned}$$

Hence,

$$S^+_{xx}[z] = \frac{2}{3} + \frac{(-5/24)}{1-(1/3)z^{-1}}.$$

Alternatively, you can use the formula to derive:

$$\alpha_k = \gamma_k \mathtt{L}[z_k^{-1}] = \begin{cases} \gamma_1 \mathtt{L}[z_1^{-1}], & k=1 \\ \gamma_2 \mathtt{L}[z_2^{-1}], & k=2 \end{cases} = \begin{cases} \gamma_1(\gamma_1+\gamma_2), & k=1 \\ \gamma_2\left(\frac{\gamma_1}{1-z_1z_2} + \frac{\gamma_2}{1-z_2^2}\right), & k=2 \end{cases} = \begin{cases} \frac{2}{3}, & k=1 \\ -\frac{5}{24}, & k=2 \end{cases}$$

which gives that

$$S^+_{xx}[z] = \sum_{i=1}^{2} \frac{\alpha_i}{1-z_i z^{-1}} = \alpha_1 + \frac{\alpha_2}{1-z_2 z^{-1}} = \frac{2}{3} + \frac{(-5/24)}{1-(1/3)z^{-1}}$$

2. (a) (10%) Find the autocorrelation function of a WSS process $\boldsymbol{x}(t)$ that is the output due to unit-power zero-mean white input $\boldsymbol{i}(t)$ and an innovation filter

$$\mathtt{L}(s) = \frac{7+5s}{(1+s)(3+s)}.$$

Hint: $\mathtt{L}(s) = \sum_{i=1}^{n} \frac{\gamma_i}{s-s_i}$ and $S^+_{xx}(s) = \sum_{i=1}^{n} \frac{\alpha_i}{s-s_i}$ and $R_{xx}(\tau) = R^+_{xx}(|\tau|)$.
where $\alpha_k = \gamma_k \mathtt{L}(-s_k)$ for $1 \le k \le n$.

(b) (10%) Repeat (a) for the innovation filter below.

$$\tilde{\mathtt{L}}(s) = \frac{7-5s}{(1-s)(3-s)}.$$

I.e., find the autocorrelation function of a WSS process $\tilde{\boldsymbol{x}}(t)$ that is the output due to unit-power zero-mean white input $\boldsymbol{i}(t)$ and the innovation filter $\tilde{\mathtt{L}}(s)$.
Hint: Think of the relation between $S_{\tilde{x}\tilde{x}}(s)$ and $\tilde{\mathtt{L}}(s)$.

Solution.

(a) With

$$\mathtt{L}(s) = \sum_{i=1}^{n} \frac{\gamma_i}{s-s_i} = \frac{1}{s+1} + \frac{4}{s+3},$$

we have $s_1 = -1$, $s_2 = -3$, $\gamma_1 = 1$, $\gamma_2 = 4$, $\alpha_1 = \gamma_1 \mathtt{L}[-s_1] = \frac{3}{2}$ and $\alpha_2 = \gamma_2 \mathtt{L}[-s_2] = \frac{11}{3}$.
Hence, from

$$\begin{aligned} R^+_{xx}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S^+_{xx}(\omega) e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\sum_{i=1}^{n} \frac{\alpha_i}{j\omega - s_i}\right) e^{j\omega\tau} d\omega \\ &= \begin{cases} \sum_{i=1}^{n} \alpha_i e^{s_i\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases} \end{aligned}$$

we have

$$R_{xx}(\tau) = R^+_{xx}(|\tau|) = \sum_{i=1}^{2} \alpha_i e^{s_i|\tau|} = \frac{3}{2}e^{-|\tau|} + \frac{11}{3}e^{-3|\tau|}$$

(b) Since $S_{\tilde{x}\tilde{x}}(s) = \tilde{\mathtt{L}}(s)\tilde{\mathtt{L}}(-s) = \mathtt{L}(-s)\mathtt{L}(s) = S_{xx}(s)$, the autocorrelation function of $\tilde{\boldsymbol{x}}(t)$ should be the same as that of $\boldsymbol{x}(t)$.

3. (a) (10%) Does the white WSS process $\boldsymbol{x}(t)$ with $S_{xx}(\omega) = \frac{N_0}{2}$ satisfy the Paley-Wiener condition?

Lemma (Paley-Wiener condition) A process $\boldsymbol{x}(t)$ is regular if the Paley-Wiener condition holds, i.e.,

$$\int_{-\infty}^{\infty} \frac{|\log S_{xx}(\omega)|}{1+\omega^2} d\omega < \infty.$$

(b) (10%) Is the Fourier transform $\boldsymbol{X}(t)$ of $\boldsymbol{x}(t)$ in (a) a regular process? A pure Yes/No answer can only earn you half of the credits. Some justification is needed in order to receive the full credits.

Definition (minimum-phase system) A system is called *minimum-phase* if both $\mathtt{L}(\omega)$ and $1/\mathtt{L}(\omega)$ are causal and have finite energy.

Definition (causal filter) A causal filter is one whose output depends only on past and present inputs.

A process that can be so represented (i.e., as a response of a *minimum-phase* system $\mathtt{L}(\omega)$ with a white input $\boldsymbol{i}(t)$ of unit power) is called *regular*.

Solution.

(a)

$$\int_{-\infty}^{\infty} \frac{|\log S_{xx}(\omega)|}{1+\omega^2} d\omega = \int_{-\infty}^{\infty} \frac{|\log(N_0/2)|}{1+\omega^2} d\omega = \pi \, |\log(N_0/2)| < \infty$$

Hence, it satisfies the Paley-Wiener condition.

(b) Since $S_{xx}(f_1, f_2) = N_0 \pi \delta(f_1 - f_2)$, we have

$$R_{XX}(\tau) = R_{XX}(u+\tau, u) = S_{xx}(u+\tau, -u) = \pi N_0 \delta(\tau).$$

Hence,

$$S_{XX}(\omega) = \pi N_0.$$

Once choice that satisfies $S_{XX}(\omega) = |\mathtt{L}(\omega)|^2$ is $\mathtt{L}(\omega) = \sqrt{\pi N_0}$, which gives $1/\mathtt{L}(\omega) = 1/\sqrt{\pi N_0}$. This indicates both $\mathtt{L}(\omega)$ and $1/\mathtt{L}(\omega)$ are causal filters. Hence, $\boldsymbol{X}(t)$ is regular.

4. (10%) Given a set of orthonormal functions $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ over $[0, T)$, define

$$\hat{\boldsymbol{x}}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \varphi_n(t) \quad \text{and} \quad \boldsymbol{c}_n = \int_0^T \boldsymbol{x}(t)\varphi_n^*(t) dt.$$

Prove that

$$E[|\hat{\boldsymbol{x}}(t) - \boldsymbol{x}(t)|^2] = 0 \quad \text{for } 0 < t < T$$

if

$$\int_0^T R_{xx}(t,s)\varphi_n(s)ds = \lambda_n\varphi_n(t)$$

for some **real** λ_n for every n, and $R_{xx}(t,t) = \sum_{n=-\infty}^{\infty} \lambda_n|\varphi_n(t)|^2$ for $0 \le t < T$.

Hint: Compute $E[|\hat{\boldsymbol{x}}(t)|^2]$ and $E[\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)]$.

Solution. Observe that

$$\begin{aligned}
E[|\hat{\boldsymbol{x}}(t)|^2] &= E\left[\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n\varphi_n(t)\right)\left(\sum_{m=-\infty}^{\infty} \boldsymbol{c}_m^*\varphi_m^*(t)\right)\right] \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E\left[\boldsymbol{c}_n\boldsymbol{c}_m^*\right]\varphi_n(t)\varphi_m^*(t) \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \lambda_m\delta[m-n]\varphi_n(t)\varphi_m^*(t) \quad \text{(because } \{c_n\} \text{ orthogonal)} \\
&= \sum_{n=-\infty}^{\infty} \lambda_n|\varphi_n(t)|^2
\end{aligned}$$

and

$$\begin{aligned}
E[\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)] &= E\left[\left(\sum_{n=-\infty}^{\infty} \boldsymbol{c}_n\varphi_n(t)\right)\boldsymbol{x}^*(t)\right] \\
&= \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n\boldsymbol{x}^*(t)]\varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} E\left[\left(\int_0^T \boldsymbol{x}(s)\varphi_n^*(s)ds\right)\boldsymbol{x}^*(t)\right]\varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} \left(\int_0^T R_{xx}(t,s)\varphi_n(s)ds\right)^*\varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} \lambda_n^*\varphi_n^*(t)\varphi_n(t) \\
&= \sum_{n=-\infty}^{\infty} \lambda_n^*|\varphi_n(t)|^2 = \sum_{n=-\infty}^{\infty} \lambda_n|\varphi_n(t)|^2 \quad (\lambda_n \text{ real and non-negative})
\end{aligned}$$

Similarly,

$$E[\hat{\boldsymbol{x}}^*(t)\boldsymbol{x}(t)] = \sum_{n=-\infty}^{\infty} \lambda_n|\varphi_n(t)|^2.$$

Hence,

$$\begin{aligned} E[|\hat{\boldsymbol{x}}(t)-\boldsymbol{x}(t)|^2] &= E[|\hat{\boldsymbol{x}}(t)|^2]-E[\hat{\boldsymbol{x}}(t)\boldsymbol{x}^*(t)]-E[\hat{\boldsymbol{x}}^*(t)\boldsymbol{x}(t)]+E[|\boldsymbol{x}(t)|^2] \\ &= \sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2-\sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2-\sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2+R_{xx}(t,t) \\ &= R_{xx}(t,t)-\sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2, \end{aligned}$$

which equals zero by

$$R_{xx}(t,t)=\sum_{n=-\infty}^{\infty}\lambda_n|\varphi_n(t)|^2.$$

□

5. (10%) Form the linear predictor of $\boldsymbol{x}[t]$ based on observation $\boldsymbol{z}[t]$ as

$$\hat{\boldsymbol{x}}[t]=\sum_{k=1}^{\infty}a_k\boldsymbol{z}[t-k],$$

and let the error process be $\boldsymbol{e}[t]=\boldsymbol{x}[t]-\hat{\boldsymbol{x}}[t]$, where $\{a_k\}_{k=1}^{\infty}$ satisfy

$$E\{\boldsymbol{e}[t]\boldsymbol{z}^*[t-m]\}=0 \text{ for any } m\geq 1.$$

Prove that $\boldsymbol{e}[t]$ is a white process.

Solution. For $\tau>0$,

$$E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{e}^*[t]\right\}=\underbrace{\cancel{E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{x}^*[t]\right\}}}_{=0}-\sum_{m=1}^{\infty}a_m\underbrace{\cancel{E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{z}^*[t-m]\right\}}}_{=0}=0.$$

For $\tau<0$,

$$E\left\{\boldsymbol{e}[t+\tau]\boldsymbol{e}^*[t]\right\}=E^*\left\{\boldsymbol{e}[t]\boldsymbol{e}^*[t+\tau]\right\}=0.$$

Hence, $\boldsymbol{e}[t]$ is white.

6. (10%) Suppose $\boldsymbol{x}(t)$ is a WSS process, and define

$$\boldsymbol{X}(u)\triangleq\int_{-\infty}^{\infty}\boldsymbol{x}(t)e^{-jut}dt.$$

Let $R_{xx}(\tau)$ and $S_{xx}(\omega)$ be the autocorrelation function and power spectrum of $\boldsymbol{x}(t)$, respectively. Also let $R_{XX}(u_1,u_2)$ and $S_{XX}(\lambda_1,\lambda_2)$ be the autocorrelation function and two-dimensional power spectrum of $\boldsymbol{X}(t)$, respectively. Then,

$$R_{XX}(u_1,u_2)=S_{xx}(u_1,-u_2)\quad\text{and}\quad S_{XX}(\lambda_1,\lambda_2)=4\pi^2R_{xx}(-\lambda_1,\lambda_2).$$

Proof:

$$\begin{aligned} R_{XX}(u_1,u_2) &= E[\boldsymbol{X}(u_1)\boldsymbol{X}^*(u_2)] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)]e^{-j(u_1t_1-u_2t_2)}dt_1dt_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(t_1,t_2)e^{-j[u_1t_1+(-u_2)t_2]}dt_1dt_2 \\ &= S_{xx}(u_1,-u_2) \end{aligned}$$

and

$$\begin{aligned} S_{XX}(\lambda_1,\lambda_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(u_1,u_2)e^{-j(\lambda_1u_1+\lambda_2u_2)}du_1du_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(u_1,-u_2)e^{-j(\lambda_1u_1+\lambda_2u_2)}du_1du_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S_{xx}(u_1,u_2')e^{j[(-\lambda_1)u_1+\lambda_2u_2']}du_1du_2' \\ &= 4\pi^2R_{xx}(-\lambda_1,\lambda_2). \end{aligned}$$

□