2012 Final Exam for Random Processes

1. (15%) Prove that if

$$\lim_{|\tau|\to\infty} C_{xx}(\tau) = 0,$$

then WSS $\boldsymbol{x}(t)$ is mean-ergodic, provided $C_{xx}(0)$ is finite.

Theorem 12-1 (Slutsky's theorem) A WSS process $\boldsymbol{x}(t)$ is mean-ergodic if, and only if,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} C_{xx}(\tau)d\tau = 0.$$

Hint: $\left|\frac{1}{2T}\int_{-T}^{T} C_{xx}(\tau)d\tau\right| \le \frac{1}{2T}\int_{-T_0}^{T_0} |C_{xx}(\tau)|d\tau + \frac{1}{2T}\int_{T_0\le|\tau|<T} |C_{xx}(\tau)|d\tau$

Solution. See slide 12-8.

2. (a) (6%) Suppose $\boldsymbol{x}(t)$ is a zero-mean white (WSS) random process with one-sided power spectrum N_0. Is $\boldsymbol{x}(t)$ mean-ergodic? Justify your answer.

   Hint: Slutsky's theorem.

   (b) (6%) Suppose $\boldsymbol{y}(t)$ is a zero-mean WSS random process with autocorrelation function $R_{yy}(\tau) = R^{+}_{yy}(|\tau|)$, where

$$R^{+}_{yy}(\tau) = \begin{cases} \sum_{i=1}^{n} \alpha_i e^{s_i\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}$$

   and each s_i is a negative real number. Is $\boldsymbol{y}(t)$ mean-ergodic? Justify your answer.

   Hint: Slutsky's theorem.

   (c) (6%) Find the best linear estimator of $\boldsymbol{y}(t)$ in (b) based on one point $\boldsymbol{y}(a)$, where $t > a$. Note that the best linear estimator minimizes the average MS error $P = E[(\boldsymbol{y}(t)-\hat{\boldsymbol{y}}(t))^2]$.

   Hint: Orthogonality principle.

   (d) (6%) Suppose $n = 1$ in (b). Then, is the random process $\boldsymbol{y}(t)$ in (b) wide-sense Markov of order 1? Justify your answer.

   Hint: A random process $\boldsymbol{s}(t)$ is named the wide-sense Markov of order 1 *if the best linear prediction based on* **one** *point is the best prediction based on the entire past. In other words, for every $\xi < a$,*

$$E[(\boldsymbol{s}(t) - \hat{\boldsymbol{s}}(t))\boldsymbol{s}(\xi)] = 0,$$

   where $\hat{\boldsymbol{s}}(t)$ is the best linear prediction based on $\boldsymbol{s}(a)$.

   (e) (6%) We further suppose that $\boldsymbol{x}(t)$ in (a) and $\boldsymbol{y}(t)$ in (b) are uncorrelated. Find the best linear estimator of $\boldsymbol{y}(t)$ in terms of $\{\boldsymbol{x}(u), a < u < b\}$.

Solution.

(a) $C_{xx}(\tau) = \frac{N_0}{2}\delta(\tau)$. So by Slutsky's theorem,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} C_{xx}(\tau)d\tau = \lim_{T\to\infty}\frac{N_0}{4T} = 0.$$

Hence, $\boldsymbol{x}(t)$ is mean-ergodic.

(b) $C_{yy}(\tau) = \sum_{i=1}^{n} \alpha_i e^{s_i|\tau|}$. So by Slutsky's theorem,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} C_{yy}(\tau)d\tau = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sum_{i=1}^{n} \alpha_i e^{s_i|\tau|} d\tau = \lim_{T\to\infty} \frac{1}{T}\sum_{i=1}^{n} \frac{\alpha_i}{-s_i}\left(1 - e^{s_i T}\right) = 0.$$

Hence, $\boldsymbol{y}(t)$ is mean-ergodic.

(c) Let $\hat{\boldsymbol{y}}(t) = h(a,t)\boldsymbol{y}(a)$. Orthogonality principle gives that

$$\begin{aligned} E[(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))\boldsymbol{y}(a)] &= R_{yy}(t-a) - h(a,t)R_{yy}(0) \\ &= \sum_{i=1}^{n} \alpha_i e^{s_i|t-a|} - h(a,t)\sum_{i=1}^{n}\alpha_i. \end{aligned}$$

Hence, the best linear estimator is

$$\hat{\boldsymbol{y}}(t) = \frac{\sum_{i=1}^{n} \alpha_i e^{s_i|t-a|}}{\sum_{i=1}^{n}\alpha_i}\boldsymbol{y}(a) = \frac{\sum_{i=1}^{n} \alpha_i e^{s_i(t-a)}}{\sum_{i=1}^{n}\alpha_i}\boldsymbol{y}(a).$$

(d) As $n = 1$, $h(a,t) = e^{s_1(t-a)}$. Hence, for $\xi < a$,

$$\begin{aligned} E[(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))\boldsymbol{y}(\xi)] &= E[(\boldsymbol{y}(t) - h(a,t)\boldsymbol{y}(a))\boldsymbol{y}(\xi)] \\ &= R_{yy}(t-\xi) - h(a,t)R_{yy}(a-\xi) \\ &= \alpha_1 e^{s_1(t-\xi)} - e^{s_1(t-a)}\alpha_1 e^{s_1(a-\xi)} \\ &= \alpha_1 e^{s_1(t-\xi)} - \alpha_1 e^{s_1(t-\xi)} \\ &= 0. \end{aligned}$$

Therefore, $\boldsymbol{y}(t)$ is wide sense Markov of order 1.

(e) Denote the best linear estimator by $\hat{\boldsymbol{y}}(t) = \int_a^b h(\alpha,t)\boldsymbol{x}(\alpha)d\alpha$. Then orthogonality principle gives that for $a < \xi < b$,

$$\begin{aligned} E[(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t))\boldsymbol{x}(\xi)] &= E\left[\left(\boldsymbol{y}(t) - \int_a^b h(\alpha,t)\boldsymbol{x}(\alpha)d\alpha\right)\boldsymbol{x}(\xi)\right] \\ &= E[\boldsymbol{y}(t)\boldsymbol{x}(\xi)] - \int_a^b h(\alpha,t)E[\boldsymbol{x}(\alpha)\boldsymbol{x}(\xi)]d\alpha \\ &= E[\boldsymbol{y}(t)]E[\boldsymbol{x}(\xi)] - \int_a^b h(\alpha,t)R_{xx}(\alpha-\xi)d\alpha \\ &= 0 - \int_a^b h(\alpha,t)\frac{N_0}{2}\delta(\alpha-\xi)d\alpha \\ &= h(\xi,t)\frac{N_0}{2} = 0 \end{aligned}$$

Hence, $h(\xi,t) = 0$ for $a < \xi < b$. As a result, the best linear estimator $\hat{\boldsymbol{y}}(t) = 0$.

3. Suppose $\boldsymbol{s}[n] = \sum_{k=0}^{\infty} \mathtt{l}[k]\boldsymbol{i}[n-k]$, where $\boldsymbol{i}[t]$ is zero-mean (WSS) white with unit variance.

(a) (6%) Find the best linear estimate $\hat{\boldsymbol{s}}[n] = \sum_{k=1}^{N} h[k,n,N]\boldsymbol{i}[n-k]$ of $\boldsymbol{s}[n]$ based on $\{\boldsymbol{i}[n-\xi], 1 \le \xi \le N\}$.

(b) (6%) What is the MS estimation error in (a)?

(c) (6%) Is the linear estimator in (a) scalable in N? Justify your answer.

(d) (6%) Is the linear estimator in (a) time invariant (with respect to time instance n)? Justify your answer.

(e) (6%) Is $\boldsymbol{s}[n]$ wide sense Markov of finite order? Justify your answer.

Hint: By "Wide sense Markov of order N", we mean that the best linear estimator based on the past N points is the best prediction based on the entire past.

Solution.

(a) Orthogonality principle gives that for $1 \le \xi \le N$:

$$\begin{aligned} E[(\boldsymbol{s}[n] - \hat{\boldsymbol{s}}[n])\boldsymbol{i}[n-\xi]] &= R_{si}[\xi] - \sum_{k=1}^{\infty} h[k,n,N]R_{ii}[\xi-k] \\ &= R_{si}[\xi] - h[\xi,n,N] = 0. \end{aligned}$$

Hence, $h[\xi,n,N] = h[\xi] = R_{si}[\xi]$. On the other hand, for $1 \le \xi \le N$,

$$R_{si}[\xi] = E[\boldsymbol{s}[n]\boldsymbol{i}[n-\xi]] = \sum_{k=0}^{\infty} \mathtt{l}[k]E\{\boldsymbol{i}[n-k]\boldsymbol{i}[n-\xi]\} = \mathtt{l}[\xi].$$

Accordingly, the best linear estimator $\hat{\boldsymbol{s}}[n]$ is given by $\hat{\boldsymbol{s}}[n] = \sum_{k=1}^{N} \mathtt{l}[k]\boldsymbol{i}[n-k]$.

(b)

$$\begin{aligned} P_N &= E[(\boldsymbol{s}[n] - \hat{\boldsymbol{s}}[n])\boldsymbol{s}[n]] \\ &= R_{ss}(0) - \sum_{k=1}^{N} \mathtt{l}[k]R_{si}[k] \\ &= \sum_{k=0}^{\infty} \mathtt{l}^2[k] - \sum_{k=1}^{N} \mathtt{l}^2[k] \\ &= \mathtt{l}^2[0] + \sum_{k=N+1}^{\infty} \mathtt{l}^2[k] \end{aligned}$$

(c) Since $h[k,n,N]$ is irrelevant to N, it is scalable in measurement size N.

(d) $h[k,n,N] = h[k]$ is nothing to do with the time instance n; so the linear estimator is time-invariant.

(e) As long as $\mathtt{l}[k] \ne 0$ infinitely often in k, the linear estimator in (a) is not wide sense Markov of finite order.

4. The modified periodogram $\boldsymbol{S}_T(\omega; c)$ of $\boldsymbol{x}(t)$ is the normalized absolute square of the Fourier transform of the known segment $\boldsymbol{x}_T(t; c) = c_T(t)\boldsymbol{x}(t)$, where $c_T(t) = c(t)p_T(t)$ satisfying $\frac{1}{2T}\int_{-T}^{T} |c_T(t)|^2 dt = 1$, and $p_T(t) \triangleq \begin{cases} 1, & |t| < T \\ 0, & |t| > T \end{cases}$, i.e.,

$$\boldsymbol{S}_T(\omega; c) = \frac{1}{2T}|\boldsymbol{X}_T(\omega; c)|^2 \quad \text{and} \quad \boldsymbol{X}_T(\omega; c) = \int_{-\infty}^{\infty} \boldsymbol{x}_T(t; c)e^{-j\omega t}dt.$$

Here, we assume that both $\boldsymbol{x}(t)$ and $c(t)$ are real.

(a) (15%) Prove that the estimation variance (namely, $E[|\boldsymbol{S}_T(\omega; c) - S_{xx}(\omega)|^2]$) is equal to the square of the bias (namely, $|E[\boldsymbol{S}_T(\omega; c)] - S_{xx}(\omega)|^2$) plus the variance of the estimate (namely, $\mathrm{Var}[\boldsymbol{S}_T(\omega; c)]$), where $\boldsymbol{S}_T(\omega; c)$ is an estimate of the spectrum $S_{xx}(\omega)$.

(b) (10%) It can be derived that

$$E[\boldsymbol{S}_T(\omega; c)] = \frac{1}{4\pi T}\int_{-\infty}^{\infty} S_{xx}(\omega - \omega')|C_T(\omega')|^2 d\omega',$$

where $C_T(\omega) \triangleq \int_{-T}^{T} c(t)e^{-j\omega t}dt$ is the Fourier transform of $c_T(t)$. Suppose

$$S_{xx}(w - v) = S_{xx}(\omega) - vS'_{xx}(\omega) + \frac{v^2}{2}S''_{xx}(\omega).$$

Show that

$$E[\boldsymbol{S}_T(\omega; c)] = S_{xx}(\omega) + \frac{S''_{xx}(\omega)}{8\pi T}\int_{-\infty}^{\infty} v^2|C_T(v)|^2 dv.$$

Hint: $C_T^*(\omega) \triangleq \int_{-T}^{T} c^*(t)e^{j\omega t}dt = \int_{-T}^{T} c(t)e^{-j(-\omega)t}dt = C_T(-\omega)$.

Solution.

(a) See slide 12-37.

(b)

$$\begin{aligned} E[\boldsymbol{S}_T(\omega; c)] &= \int_{-\infty}^{\infty} S_{xx}(\omega - v) \cdot \frac{1}{4\pi T}|C_T(v)|^2 dv \\ &= \int_{-\infty}^{\infty} S_{xx}(\omega) \cdot \frac{1}{4\pi T}|C_T(v)|^2 dv - \int_{-\infty}^{\infty} vS'_{xx}(\omega) \cdot \frac{1}{4\pi T}|C_T(v)|^2 dv \\ &\quad + \int_{-\infty}^{\infty} \frac{v^2}{2}S''_{xx}(\omega) \cdot \frac{1}{4\pi T}|C_T(v)|^2 dv \\ &= S_{xx}(\omega) + \frac{S''_{xx}(\omega)}{8\pi T}\int_{-\infty}^{\infty} v^2|C_T(v)|^2 dv, \end{aligned}$$

where the last step follows from $|C_T(v)| = |C_T(-v)|$.