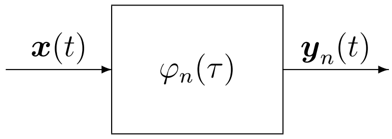
2007 Final Exam for Random Processes



1. (a) (8 pt.) Let $\{\varphi_n(\tau)\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ satisfy that

$$\int_{-\infty}^{\infty} \varphi_n(\tau)\varphi_m^*(\tau)d\tau = \delta[n-m]$$

and

$$\int_{-\infty}^{\infty} R_{xx}(t-s)\varphi_n(s)ds = \lambda_n \varphi_n(t),$$

where $R_{xx}(\tau)$ is the autocorrelation function of WSS process $\boldsymbol{x}(t)$, and $\delta[\cdot]$ is the Kronecker delta function. Let $\boldsymbol{y}_n(t)$ be the output process through the linear time-invariant filter $\varphi_n(\tau)$ due to input process $\boldsymbol{x}(t)$ as shown in the above figure. Prove that $E[\boldsymbol{y}_n(t)\boldsymbol{y}_m^*(t)] = \lambda_n \delta[n-m]$.

(b) (6 pt.) Prove that

$$\int_{-\infty}^{\infty} R_{xx}(t-s)\varphi(s)ds = C \cdot \varphi(t)$$

is valid for any continuous function $\varphi(\tau)$, provided that $\boldsymbol{x}(t)$ is white and WSS with autocorrelation function $R_{xx}(\tau) = C \cdot \delta(\tau)$.

(c) (6 pt.) By following (b), what is the eigenvalue for a white noise with two-sided power spectrum density $N_0/2$?

Solution.

(a)

$$\begin{aligned}
E[\boldsymbol{y}_n(t)\boldsymbol{y}_m^*(t)] &= E\left[\left(\int_{-\infty}^{\infty} \varphi_n(u)\boldsymbol{x}(t-u)du\right)\left(\int_{-\infty}^{\infty} \varphi_m(v)\boldsymbol{x}(t-v)dv\right)^*\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_n(u)\varphi_m^*(v)E\left[\boldsymbol{x}(t-u)\boldsymbol{x}^*(t-v)\right]dudv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_n(u)\varphi_m^*(v)R_{xx}(v-u)dudv \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} R_{xx}(v-u)\varphi_n(u)du\right)\varphi_m^*(v)dv \\
&= \int_{-\infty}^{\infty} \lambda_n\varphi_n(v)\varphi_m^*(v)dv \\
&= \lambda_n\delta[n-m].
\end{aligned}$$

(b) By the replication property,

$$\begin{aligned}\int_{-\infty}^{\infty} R_{xx}(t-s)\varphi(s)ds &= \int_{-\infty}^{\infty} C\cdot\delta(t-s)\varphi(s)ds \\ &= C\varphi(t).\end{aligned}$$

(c) $N_0/2$. □

2. (8 pt.) Prove that if $\boldsymbol{x}(t)$ is zero-mean, white and WSS, then $\boldsymbol{X}(\omega)=\int_{-\infty}^{\infty}\boldsymbol{x}(t)e^{-j\omega t}dt$ is also zero-mean, white and WSS.

Solution. It is obvious that $\boldsymbol{X}(\omega)$ is zero-mean; hence, we omit its proof. Continuing from the proof in slide 11-73, we derive that the autocorrelation function of $\boldsymbol{X}(\omega)$ is

$$R_{XX}(u,v)=2\pi S_{xx}(u)\delta(u-v)=2\pi C\delta(u-v),$$

where $S_{xx}(u)=C$ for some C, by the assumption of whiteness. Hence, $\boldsymbol{X}(\omega)$ is also a white process. By the way, the autocorrelation of $\boldsymbol{X}(\omega)$ only depends on the frequency-difference; hence, it is WSS since it is zero-mean. □

3. (a) (8 pt.) Is there a continuous-in-time process that is simultaneously regular and predictable? Justify your answer with a formal proof. (Hint: A process is predictable if, and only, if it has non-degenerate line spectra of countably many lines.)

   (b) (8 pt.) Let $\hat{\boldsymbol{x}}[t]=\sum_{k=1}^{\infty}a_k\boldsymbol{x}[t-k]$ be the best linear MS estimator of discrete-in-time non-predictable WSS process $\boldsymbol{x}[t]$ based upon its past. Find a whitening filter of $\boldsymbol{x}[t]$. (Hint: The answer is inside the proof of the Wold's decomposition.)

   (c) (8 pt.) Does (b) imply that all WSS processes can be whitened by a linear filter? Justify your answer. (Hint: A whitening filter of process $\boldsymbol{x}[t]$, by definition, shall make the output becoming a white process with power spectrum of "non-zero" height. In other words, "nullification" (zero-output) cannot be considered as a whitening procedure.)

Solution.

(a) Suppose $\boldsymbol{x}(t)$ is both regular and predictable. Then, $S_{xx}(\omega)=|\mathtt{L}(\omega)|^2$ for some causal filter $\mathtt{L}(\omega)$. Since $\boldsymbol{x}(t)$ is also predictable, $|\mathtt{L}(\omega)|^2$ only consists of countably many impulses, i.e., $\mathtt{L}(\omega)=\sum_n c_n\delta(\omega-\omega_n)$ for complex numbers $\{c_n\}$. The impulse response of $\mathtt{L}(\omega)$ is therefore equal to $\frac{1}{2\pi}\sum_n c_n e^{j\omega_n\tau}$, which is not zero for $\tau<0$ a violation to causality. Hence, such a process does not exist.

(b) By the proof of the Wold's decomposition, $\boldsymbol{e}[t]=\boldsymbol{x}[t]-\hat{\boldsymbol{x}}[t]$ is the output due to input $\boldsymbol{x}[t]$ through filter $A[z]=1-\sum_{k=1}^{\infty}a_k z^{-k}$, and $\{\boldsymbol{e}[t]\}$ is white. Hence, $A[z]$ is a whitening filter of $\boldsymbol{x}[t]$.

(c) By Wold's decomposition, $\boldsymbol{x}[t] = \boldsymbol{x}_p[t] + \boldsymbol{x}_r[t]$, where $\boldsymbol{x}_p[t]$ is predictable, and $\boldsymbol{x}_r[t]$ is regular. From the proof, we learn that $\boldsymbol{x}_p[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}_p[t-k] = \boldsymbol{y}[t] = 0$ with probability one. Hence,

$$\begin{aligned}\boldsymbol{x}[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}[t-k] &= \left(\boldsymbol{x}_p[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}_p[t-k]\right) + \left(\boldsymbol{x}_r[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}_r[t-k]\right) \\ &= \boldsymbol{x}_r[t] - \sum_{k=1}^{\infty} a_k \boldsymbol{x}_r[t-k].\end{aligned}$$

Hence, it seems that all processes can be whitened by its corresponding $A[z]$. However, if $\boldsymbol{x}[t]$ is predictable itself, then the output $\boldsymbol{x}[t] - \hat{\boldsymbol{x}}[t]$ will be zero. In such case, $A[z]$ cannot be claimed to be the whitening filter of $\boldsymbol{x}[t]$. □

4. (a) (8 pt.) Prove that a discrete-in-time zero-mean white process $\boldsymbol{i}[t]$ of finite power is mean-ergodic in the sense that

$$E\left[|\boldsymbol{\eta}_T - \eta|^2\right] \to 0 \text{ as } T \to \infty$$

for some η, where

$$\boldsymbol{\eta}_T \triangleq \frac{1}{2T+1} \sum_{t=-T}^{T} \boldsymbol{i}[t].$$

(b) (8 pt.) By following (a), prove that any process of the form $\boldsymbol{x}[t] = \sum_{k=0}^{n} h_k \boldsymbol{i}[t-k]$ is mean-ergodic, provided that $\sum_{k=0}^{n} |h_k| < \infty$.

(c) (6 pt.) Answer without proof that whether a regular process is mean-ergodic.

Solution.

(a) For $\eta = 0$,

$$E\left[|\boldsymbol{\eta}_T|^2\right] = \frac{1}{2T+1} E[|\boldsymbol{i}[t]|^2] \to 0.$$

(b)

$$\begin{aligned}\boldsymbol{\eta}_T &= \frac{1}{2T+1} \sum_{t=-T}^{T} \sum_{k=0}^{n} h_k \boldsymbol{i}[t-k] \\ &= \frac{1}{2T+1} \sum_{u=-T-n}^{T} w_u \boldsymbol{i}[u],\end{aligned}$$

where $|w_u| \le \sum_{k=0}^{n} |h_k| \triangleq w_{\max}$. Hence,

$$E\left[|\boldsymbol{\eta}_T|^2\right] \le \frac{|w_{\max}|^2}{2T+1} E[|\boldsymbol{i}[t]|^2] \to 0.$$

(c) It follows from (b) that a regular process should be mean-ergodic. □

5. (a) (8 pt.) Prove that "estimation variance" is equal to the sum of "variance of estimation" and the square of the bias. In other words,

$$E\left[|g(\boldsymbol{x}) - x|^2\right] = E\left[|g(\boldsymbol{x}) - E[g(\boldsymbol{x})]|^2\right] + |E[g(\boldsymbol{x})] - x|^2,$$

where $g(\boldsymbol{x})$ is an estimator of x based on the random variable $\boldsymbol{x}$. (Note that x is in general a complex number, and $\boldsymbol{x}$ is a complex random variable.)

(b) (8 pt.) By following (a), the linear estimator of x based on the observation $\boldsymbol{x}$ is of the form $g(\boldsymbol{x}) = c \cdot \boldsymbol{x}$ for some complex constant c. Find the best c that minimizes the estimation variance. (Hint: You may denote $\mu_x = E[\boldsymbol{x}]$ and $\sigma_x^2 = E[|\boldsymbol{x} - \mu_x|^2]$ for convenience.)

Solution.

(a) For convenience, abbreviate $g = g(\boldsymbol{x})$ and $\mu = E[g(\boldsymbol{x})]$ in my solution.

$$\begin{aligned}
& E\left[|g(\boldsymbol{x}) - E[g(\boldsymbol{x})]|^2\right] + |E[g(\boldsymbol{x})] - x|^2 \\
= \ & E\left[|g - \mu|^2\right] + |\mu - x|^2 \\
= \ & E[gg^*] - \mu E[g^*] - \mu^* E[g] + \mu\mu^* \\
+ \ & \mu\mu^* - x\mu^* - x^*\mu + xx^* \\
= \ & E[gg^*] - \mu\mu^* - \mu^*\mu + \mu\mu^* \\
+ \ & \mu\mu^* - x\mu^* - x^*\mu + xx^* \\
= \ & E[|g - x|^2].
\end{aligned}$$

(b) Denote the mean and variance of $\boldsymbol{x}$ by $E[\boldsymbol{x}] = \mu_x$ and $E[|\boldsymbol{x} - \mu_x|^2] = \sigma_x^2$. Then,

$$|\text{bias}|^2 = |c\mu_x - x|^2 = |c|^2|\mu_x|^2 - c^*\mu_x^* x - c\mu_x x^* + |x|^2$$

and

$$\text{variance of estimate} = |c|^2\sigma_x^2.$$

Hence, the estimation variance is given by:

$$\begin{aligned}
& |c|^2(\sigma_x^2 + |\mu_x|^2) - c\mu_x x^* - xc^*\mu_x^* + |x|^2 \\
= \ & (\sigma_x^2 + |\mu_x|^2)\left(|c|^2 - c\frac{\mu_x x^*}{\sigma_x^2 + |\mu_x|^2} - c^*\frac{x\mu_x^*}{\sigma_x^2 + |\mu_x|^2}\right) + |x|^2 \\
= \ & (\sigma_x^2 + |\mu_x|^2)\left|c - \frac{x\mu_x^*}{\sigma_x^2 + |\mu_x|^2}\right|^2 + \frac{\sigma_x^2}{\sigma_x^2 + |\mu_x|^2}|x|^2.
\end{aligned}$$

Hence, the best linear estimator is $g(\boldsymbol{x}) = \frac{x\mu_x^*}{\sigma_x^2 + |\mu_x|^2}\boldsymbol{x}$.

6. (10 pt.) Can we find a real data window $c(t)$ to exactly emulate the effect of the real spectral window $W(y)$ for **all** input processes? Justify your answer. (Hint:

$$\boldsymbol{S}_{T,w}(\omega;c) = \frac{1}{4\pi T} \left|\boldsymbol{X}_T(\omega) * C(\omega)\right|^2 * W(\omega),$$

where $C(\omega)$ is the Fourier transform of $c(t)$, and "$*$" denotes the convolution operation.)

Solution. The problem is equivalent to finding $C(\omega)$ (for a given $W(\omega)$) such that

$$\left|\boldsymbol{X}_T(\omega) * C(\omega)\right|^2 = \left|\boldsymbol{X}_T(\omega)\right|^2 * W(\omega).$$

The right-hand-side is a system with input $\boldsymbol{X}_T(\omega)$ and output $\left|\boldsymbol{X}_T(\omega)\right|^2 * W(\omega)$, and the left-hand-side is a system with input $\boldsymbol{X}_T(\omega)$ and output $\left|\boldsymbol{X}_T(\omega) * C(\omega)\right|^2$. If the two systems are equivalent, then the outputs should be the same for any input $\boldsymbol{X}_T(\omega)$. Yet, the right-hand-side system is irrelevant to the phase of the complex $\boldsymbol{X}(\omega)$, while the left-hand-side is a function of the phase of the complex $\boldsymbol{X}(\omega)$. So, no matter how we design $C(\omega)$, there always exists a specific input process to fail the equivalence. So, the answer of the problem is NO.