2007 First Midterm for Random Processes

1. (16 pt.) Please define a non-deterministic SSS random process $\{\boldsymbol{y}(t), t \in \Re\}$ over the probability space $(S = \{\oplus, \ominus, \otimes, \oslash\}, \mathcal{F} = 2^S, P = \{0.1, 0.2, 0.3, 0.4\}$ resp. for $S)$. Please justify your answer.
(Hint: If the input $\boldsymbol{x}(t)$ to a memoryless system is SSS, its output $\boldsymbol{y}(t)$ is also SSS.)
(Note: You should justify both the SSS property and the non-deterministic nature of your process in the answer.)

Solution. Define a random process $\{\boldsymbol{x}(t), t \in \Re\}$ over the given $(S, \mathcal{F}, P)$ as $\boldsymbol{x}(t, \zeta) = 1$ for all $\zeta \in S$ and all $t \in \Re$. This constant process is surely SSS.

Define the memoryless system (random variable) $\{\boldsymbol{T}(x), x \in \{1\}\}$ over the given $(S, \mathcal{F}, P)$ as

$$\boldsymbol{T}(1, \oplus) = 1, \quad \boldsymbol{T}(1, \ominus) = -1, \quad \boldsymbol{T}(1, \otimes) = 1 \quad \text{and} \quad \boldsymbol{T}(1, \oslash) = -1.$$

$$\text{Then, } \begin{cases} \boldsymbol{y}(t, \oplus) &= \boldsymbol{T}(\boldsymbol{x}(t, \oplus), \oplus) = \boldsymbol{T}(1, \oplus) = 1 \\ \boldsymbol{y}(t, \ominus) &= \boldsymbol{T}(\boldsymbol{x}(t, \ominus), \ominus) = \boldsymbol{T}(1, \ominus) = -1 \\ \boldsymbol{y}(t, \otimes) &= \boldsymbol{T}(\boldsymbol{x}(t, \otimes), \otimes) = \boldsymbol{T}(1, \otimes) = 1 \\ \boldsymbol{y}(t, \oslash) &= \boldsymbol{T}(\boldsymbol{x}(t, \oslash), \oslash) = \boldsymbol{T}(1, \oslash) = -1 \end{cases}$$

Justification one. By the lemma that the output process due to the input process through a memoryless system $\boldsymbol{T}$ is SSS, the SSS of the above defined random process $\{\boldsymbol{y}(t), t \in \Re\}$ is justified!

Justification two. Since

$$\Pr\{\boldsymbol{y}(t) = 1\} = P\left(\{\zeta \in S : \boldsymbol{y}(t, \zeta) = 1\}\right) = P\left(\{\oplus, \otimes\}\right) = 0.1 + 0.3 = 0.4,$$

the non-deterministic nature of the process $\boldsymbol{y}(t)$ is justified! □

2. Denote by $f(x_1, x_2; t_1, t_2)$ the two-dimensional joint density of samples $\boldsymbol{x}(t_1)$ and $\boldsymbol{x}(t_2)$ of *stationary* real-valued random process $\{\boldsymbol{x}(t), t \in \Re\}$, where each $\boldsymbol{x}(t) \in \Re$. Let $q(x_2; t_1, t_2) \triangleq \int_{-\infty}^{\infty} x_1 \cdot f(x_1, x_2; t_1, t_2) dx_1$. Define a (statistical) memoryless system $\boldsymbol{T}$.

   (a) (5 pt.) Show that $R_{xx}(t_1, t_2) = \int_{-\infty}^{\infty} x_2 \cdot q(x_2; t_1, t_2) dx_2$.

   (b) (5 pt.) Show that $R_{xy}(t_1, t_2) = \int_{-\infty}^{\infty} g(x_2) \cdot q(x_2; t_1, t_2) dx_2$, where $\{\boldsymbol{y}(t), t \in \Re\}$ is the output process due to input $\boldsymbol{x}(t)$ and memoryless system $\boldsymbol{T}$, and $g(x_2) = E[\boldsymbol{T}(x_2)]$.

   (c) (5 pt.) If $\boldsymbol{x}(t_1)$ and $\boldsymbol{x}(t_2)$ are independent for any $t_1 \neq t_2$ with densities $f(\cdot; t_1)$ and $f(\cdot; t_2)$, respectively, is $R_{xy}(t_1, t_2)/R_{xx}(t_1, t_2)$ a constant, independent of t_1 and t_2?

Solution.

(a)

$$\begin{aligned}
R_{xx}(t_1,t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2 f(x_1,x_2;t_1,t_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty} x_2\left(\int_{-\infty}^{\infty} x_1 f(x_1,x_2;t_1,t_2)dx_1\right)dx_2 \\
&= \int_{-\infty}^{\infty} x_2\cdot q(x_2;t_1,t_2)dx_2
\end{aligned}$$

(b)

$$\begin{aligned}
R_{xy}(t_1,t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{y}^*(t_2)] \\
&= E[\boldsymbol{x}(t_1)\cdot \boldsymbol{T}(\boldsymbol{x}(t_2))] \\
&= E\left[E\left[x_1\cdot \boldsymbol{T}(x_2)\right|\boldsymbol{x}(t_1)=x_1,\boldsymbol{x}(t_2)=x_2]\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_1\, y\, dP_{\boldsymbol{T}}(y|x_2)\right) f(x_1,x_2;t_1,t_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 g(x_2) f(x_1,x_2;t_1,t_2)dx_1dx_2 \\
&= \int_{-\infty}^{\infty} g(x_2)\cdot q(x_2;t_1,t_2)dx_2.
\end{aligned}$$

(c) If $\boldsymbol{x}(t_1)$ and $\boldsymbol{x}(t_2)$ are independent for any $t_1 \neq t_2$ with respectively density $f(\cdot;t_1)$ and $f(\cdot;t_2)$, then

$$\begin{aligned}
q(x_2;t_1,t_2) &= \int_{-\infty}^{\infty} x_1\cdot f(x_1,x_2;t_1,t_2)dx_1 \\
&= \int_{-\infty}^{\infty} x_1\cdot f(x_1;t_1)f(x_2;t_2)dx_1 \\
&= E[\boldsymbol{x}(t_1)]f(x_2;t_2).
\end{aligned}$$

Hence,

$$R_{xx}(t_1,t_2) = \int_{-\infty}^{\infty} x_2 f(x_2;t_2)E[\boldsymbol{x}(t_1)]dx_2 = E[\boldsymbol{x}(t_2)]E[\boldsymbol{x}(t_1)].$$

and

$$R_{xy}(t_1,t_2) = \int_{-\infty}^{\infty} g(x_2)\cdot E[\boldsymbol{x}(t_1)]f(x_2;t_2)dx_2 = E[g(\boldsymbol{x}(t_2))]E[\boldsymbol{x}(t_1)].$$

As a result,

$$\frac{R_{xy}(t_1,t_2)}{R_{xx}(t_1,t_2)} = \frac{E[g(\boldsymbol{x}(t_2))]}{E[\boldsymbol{x}(t_2)]}.$$

Since $\boldsymbol{x}(t)$ is stationary, then $\frac{E[g(\boldsymbol{x}(t_2))]}{E[\boldsymbol{x}(t_2)]}$ becomes a constant. $\square$

3. (a) (5 pt.) Give an example that a (non-deterministic) WSS input to a memoryless system does not produce a WSS output. You should justify your answer.

   (b) (5 pt.) Give an example of a linear system in which the output cannot be represented as the convolution of the input process and system impulse response. You should justify your answer.

   (c) (5 pt.) Give an example of a random process, whose autocorrelation function only depends on the time difference, but whose mean function is not a constant.

   (d) (5 pt.) Give an example of a complex WSS random process, whose power density spectrum is equal to the given non-negative integrable function $S(\omega)$.

   (e) (4 pt.) From lectures, it is known that a function $R(\tau)$ has non-negative Fourier transform if, and only if, it is p.d., i.e.,

$$\sum_{i,j} a_i a_j^* R(t_i - t_j) \geq 0 \quad \text{for any complex } a_i \text{ and } a_j.$$

   Now, can a non-p.d. function has non-negative Fourier transform? Just answer with Yes or No.

Solution.

(a) Slide 9-65.

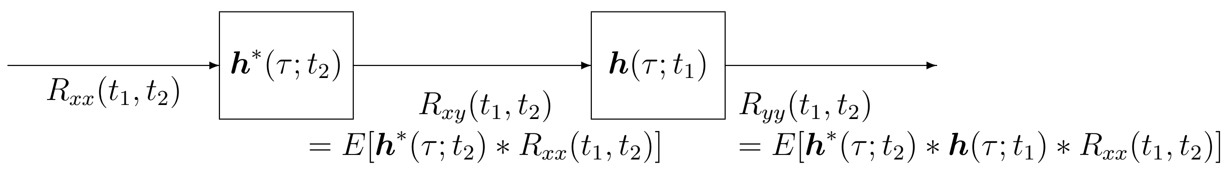
(b) Show that a differentiator has no legitimate impulse response, but it is a linear system.

(c) Slide 9-91.

(d) Slide 9-90.

(e) No.

4. (16 pt.) Prove that for any linear system, whose input and output relationship can be characterized by convolution operation, $R_{xy}(t_1, t_2) = E[\boldsymbol{h}^*(\tau; t_2) * R_{xx}(t_1, t_2)]$ and $R_{yy}(t_1, t_2) = E[\boldsymbol{h}^*(\tau; t_2) * \boldsymbol{h}(\tau; t_1) * R_{xx}(t_1, t_2)]$.



Solution.

$$\begin{aligned}
R_{xy}(t_1,t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{y}^*(t_2)] \\
&= E\left[\boldsymbol{x}(t_1)\int_{-\infty}^{\infty}\boldsymbol{h}^*(\tau;t_2)\boldsymbol{x}^*(t_2-\tau)d\tau\right] \\
&= \int_{-\infty}^{\infty} E[\boldsymbol{h}^*(\tau;t_2)]E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2-\tau)]d\tau \\
&= \int_{-\infty}^{\infty} E[\boldsymbol{h}^*(\tau;t_2)]R_{xx}(t_1,t_2-\tau)d\tau \\
&= E\left[\int_{-\infty}^{\infty}\boldsymbol{h}^*(\tau;t_2)R_{xx}(t_1,t_2-\tau)d\tau\right].
\end{aligned}$$

$$\begin{aligned}
R_{yy}(t_1,t_2) &= E[\boldsymbol{y}(t_1)\boldsymbol{y}^*(t_2)] \\
&= E\left[\int_{-\infty}^{\infty}\boldsymbol{h}(\tau;t_1)\boldsymbol{x}(t_1-\tau)d\tau\int_{-\infty}^{\infty}\boldsymbol{h}^*(s;t_2)\boldsymbol{x}^*(t_2-s)ds\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\boldsymbol{h}^*(s;t_2)\boldsymbol{h}(\tau;t_1)]E[\boldsymbol{x}(t_1-\tau)\boldsymbol{x}^*(t_2-s)]d\tau ds \\
&= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\boldsymbol{h}^*(s;t_2)\boldsymbol{h}(\tau;t_1)R_{xx}(t_1-\tau,t_2-s)d\tau ds\right].
\end{aligned}$$

5. (12 pt.) A process $\boldsymbol{x}(t)$ is called *MS periodic* if

$$E[|\boldsymbol{x}(t+T)-\boldsymbol{x}(t)|^2]=0$$

for every t. Based on the definition, prove that a process $\boldsymbol{x}(t)$ is *MS periodic* if, and only if, its autocorrelation function is *doubly periodic*, namely,

$$R_{xx}(t_1+mT,t_2+nT)=R_{xx}(t_1,t_2) \text{ for every integer } m \text{ and } n.$$

Solution. Please refer slide 9-108 and 9-109. □

6. (12 pt.) Prove that a discrete process $\{\boldsymbol{x}[n]=\sum_{i=1}^{\infty}\boldsymbol{c}_i e^{j\omega_i n}, n\in\{1,2,3,\cdots\}\}$ is WSS if, and only if, $\{\boldsymbol{c}_i\}_{i=1}^{\infty}$ are uncorrelated with zero mean.

Solution. Please refer slide 9-123. □