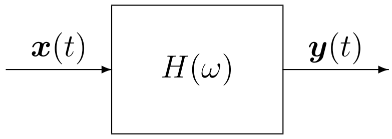
2007 Second Midterm for Random Processes



1. Given that $R_{xx}(\tau) = R_{yy}(\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$, and that real random process $\boldsymbol{x}(t)$ and real random process $\boldsymbol{y}(t)$ are joint WSS, prove that the stable linear time-invariant filter $H(\omega)$ in the above figure should satisfy:

   (a) $|H(\omega)|^2 = 1$;

   (b) $H^*(\omega) = -H(\omega)$;

   (c) $H(-\omega) = -H(\omega)$.

   (Hint: $S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$, $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$ and $S_{xx}(\omega) = S_{xx}(-\omega)$.)

   Solution.

   (a) $R_{xx}(\tau) = R_{yy}(\tau)$ implies that $S_{xx}(\omega) = S_{yy}(\omega)$, which, together with $S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2$, indicates $|H(\omega)|^2 = 1$.

   (b) $R_{xy}(\tau) = -R_{yx}(\tau)$ implies that $S_{xy}(\omega) = -S_{yx}(\omega)$, which, together with $S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega)$ and $S_{yx}(\omega) = S_{yy}(\omega)/H^*(\omega) = S_{xx}(\omega)|H(\omega)|^2/H^*(\omega) = S_{xx}(\omega)H(\omega)$, indicates $H^*(\omega) = -H(\omega)$.

   (c) $R_{xy}(\tau) = E[\boldsymbol{x}(t+\tau)\boldsymbol{y}(t)] = E[\boldsymbol{y}(t)\boldsymbol{x}(t+\tau)] = R_{yx}(-\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$ imply $R_{xy}(\tau) = -R_{xy}(-\tau)$, which in turn implies $S_{xy}(\omega) = -S_{xy}(-\omega)$. Hence, $S_{xx}(\omega)H^*(\omega) = -S_{xx}(-\omega)H^*(-\omega) = -S_{xx}(\omega)H^*(-\omega)$. This completes the proof that $H(\omega) = -H(-\omega)$. □

2. (a) For a given (WSS) random process $\boldsymbol{z}(t)$ with power spectrum density $S_{zz}(\omega)$, determine the optimal ω_0 that minimizes $E[|\boldsymbol{w}'(t)|^2]$ provided that $\boldsymbol{w}(t) \triangleq \boldsymbol{z}(t)e^{-j\omega_0 t}$.

   (b) If $\boldsymbol{z}(t) = e^{j(\omega_1 t + \lambda\boldsymbol{\varphi}(t))}$ for some SSS random process $\boldsymbol{\varphi}(t)$, satisfying that $\boldsymbol{\varphi}(0) = 0$. Let the probability density of $\boldsymbol{\varphi}(t)$ be $f_{\boldsymbol{\varphi}}(\cdot)$. Determine the optimal ω_0 in (a).

   (c) Repeat (b) if $\boldsymbol{z}(t) = e^{j\boldsymbol{\omega}t}$ for some random variable $\boldsymbol{\omega}$ with probability density $f_{\boldsymbol{\omega}}(\cdot)$.

   (Hint: For a linear system with impulse response $h(\tau;t) = h_1(\tau)e^{j\omega_0 t}$ and WSS input, $S_{yy}(\omega) = S_{xx}(\omega-\omega_0)|H(\omega-\omega_0)|^2$, where $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are the input and output processes, respectively.)

   Solution.

(a) From the hint, $S_{ww}(\omega) = S_{zz}(\omega + \omega_0)$. Also, $S_{w'w'}(\omega) = S_{ww}(\omega)|j\omega|^2 = S_{ww}(\omega)\omega^2$. Therefore,

$$\begin{aligned}
E[|\boldsymbol{w}'(t)|^2] &= R_{w'w'}(0) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{w'w'}(\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ww}(\omega)\omega^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{zz}(\omega+\omega_0)\omega^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} (u-\omega_0)^2 S_{zz}(u)du, \quad u = \omega + \omega_0 \\
&= \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} u^2 S_{zz}(u)du - 2\omega_0\int_{-\infty}^{\infty} uS_{zz}(u)du + \omega_0^2\int_{-\infty}^{\infty} S_{zz}(u)du\right).
\end{aligned}$$

Taking the derivative of $E[|\boldsymbol{w}'(t)|^2]$ with respect to ω_0 yields that

$$\frac{dE[|\boldsymbol{w}'(t)|^2]}{d\omega_0} = \frac{1}{2\pi}\left(-2\int_{-\infty}^{\infty} uS_{zz}(u)du + 2\omega_0\int_{-\infty}^{\infty} S_{zz}(u)du\right) = 0,$$

which implies

$$\omega_0 = \frac{\int_{-\infty}^{\infty} uS_{zz}(u)du}{\int_{-\infty}^{\infty} S_{zz}(u)du}.$$

(b)

$$\begin{aligned}
R_{zz}(\tau) &= E[\boldsymbol{z}(t+\tau)\boldsymbol{z}^*(t)] \\
&= E\left[e^{j(\omega_1(t+\tau)+\lambda\boldsymbol{\varphi}(t+\tau))}e^{-j(\omega_1 t+\lambda\boldsymbol{\varphi}(t))}\right] \\
&= E\left[e^{j(\omega_1\tau+\lambda\boldsymbol{\varphi}(\tau))}\right] \quad (\boldsymbol{\varphi}(t+\tau)-\boldsymbol{\varphi}(t) \equiv \boldsymbol{\varphi}(\tau)-\boldsymbol{\varphi}(0) \equiv \boldsymbol{\varphi}(\tau)) \\
&= e^{j\omega_1\tau}\int_{-\infty}^{\infty} f_{\boldsymbol{\varphi}}(u)e^{j\lambda u}du
\end{aligned}$$

Then,

$$\begin{aligned}
S_{zz}(\omega) &= \int_{-\infty}^{\infty} R_{zz}(\tau)e^{-j\omega\tau}d\tau \\
&= \int_{-\infty}^{\infty} e^{j\omega_1\tau}\left(\int_{-\infty}^{\infty} f_{\boldsymbol{\varphi}}(u)e^{j\lambda u}du\right)e^{-j\omega\tau}d\tau \\
&= \left(\int_{-\infty}^{\infty} e^{-j(\omega-\omega_1)\tau}d\tau\right)\left(\int_{-\infty}^{\infty} f_{\boldsymbol{\varphi}}(u)e^{j\lambda u}du\right) \\
&= 2\pi\delta(\omega-\omega_1)\left(\int_{-\infty}^{\infty} f_{\boldsymbol{\varphi}}(u)e^{j\lambda u}du\right).
\end{aligned}$$

Accordingly,

$$\omega_0 = \frac{\int_{-\infty}^{\infty} u S_{zz}(u) du}{\int_{-\infty}^{\infty} S_{zz}(u) du} = \frac{\int_{-\infty}^{\infty} u \delta(u-\omega_1) du}{\int_{-\infty}^{\infty} \delta(u-\omega_1) du} = \omega_1.$$

(c)

$$\begin{aligned} R_{zz}(\tau) &= E[\boldsymbol{z}(t+\tau)\boldsymbol{z}^*(t)] \\ &= E\left[e^{j\boldsymbol{\omega}(t+\tau)} e^{-j\boldsymbol{\omega} t}\right] \\ &= E\left[e^{j\boldsymbol{\omega}\tau}\right] \\ &= \int_{-\infty}^{\infty} f_{\boldsymbol{\omega}}(\omega) e^{j\omega\tau} d\omega \end{aligned}$$

Then,

$$\begin{aligned} S_{zz}(\omega) &= \int_{-\infty}^{\infty} R_{zz}(\tau) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_{\boldsymbol{\omega}}(u) e^{ju\tau} du \right) e^{-j\omega\tau} d\tau \\ &= \int_{-\infty}^{\infty} f_{\boldsymbol{\omega}}(u) \left(\int_{-\infty}^{\infty} e^{-j(\omega-u)\tau} d\tau \right) du \\ &= \int_{-\infty}^{\infty} f_{\boldsymbol{\omega}}(u) 2\pi \delta(\omega-u) du \\ &= 2\pi f_{\boldsymbol{\omega}}(\omega). \end{aligned}$$

Accordingly,

$$\omega_0 = \frac{\int_{-\infty}^{\infty} u S_{zz}(u) du}{\int_{-\infty}^{\infty} S_{zz}(u) du} = E[\boldsymbol{\omega}].$$

□

3. (a) For a system with $\boldsymbol{y}(t) = \boldsymbol{a}(t) \cdot x(t)$, where $x(t)$ and $\boldsymbol{y}(t)$ respectively represent the deterministic input and random output of the system, and $\boldsymbol{a}(t)$ is WSS, prove that

$$\frac{1}{E[\boldsymbol{a}(t)]} \boldsymbol{Y}(\omega)$$

is an unbiased estimator of the Fourier transform $X(\omega)$ of $x(t)$. (Note that $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ and $\boldsymbol{Y}(\omega) = \int_{-\infty}^{\infty} \boldsymbol{y}(t) e^{-j\omega t} dt$.)

(b) Find the unbiased estimator of $X(\omega)$ if $\boldsymbol{a}(t) = \sum_{n=-\infty}^{\infty} \delta(t - \boldsymbol{t}_n)$, where $\boldsymbol{a}(t)$ is WSS with $E[\boldsymbol{a}(t)] = 1$.

Solution.

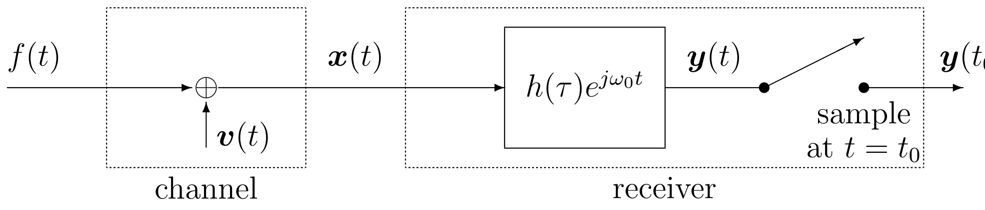
(a)

$$\begin{aligned} E[\boldsymbol{Y}(\omega)] &= E\left[\int_{-\infty}^{\infty} \boldsymbol{y}(t)e^{-j\omega t}dt\right] \\ &= \int_{-\infty}^{\infty} E[\boldsymbol{y}(t)]e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)E[\boldsymbol{a}(t)]e^{-j\omega t}dt \\ &= E[\boldsymbol{a}(t)]X(\omega). \end{aligned}$$

(b)

$$\begin{aligned} \boldsymbol{Y}(\omega) &= \int_{-\infty}^{\infty} \boldsymbol{y}(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} \left[x(t)\sum_{n=-\infty}^{\infty} \delta(t-\boldsymbol{t}_n)\right] e^{-j\omega t}dt \\ &= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\delta(t-\boldsymbol{t}_n)e^{-j\omega t}dt \\ &= \sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n)e^{-j\omega \boldsymbol{t}_n}. \end{aligned}$$

□



4. According to the above figure, determine the filter $h(\tau)$ such that the *output signal-to-noise ratio* is maximized, provided $\boldsymbol{v}(t)$ is WSS with power spectrum density $S_{vv}(\omega)$. Will the value of ω_0 affect the resultant optimal filter?

Solution. Note that

$$y_f(t) = \int_{-\infty}^{\infty} h(\tau;t)f(t-\tau)d\tau = \int_{-\infty}^{\infty} \left(h(\tau)e^{j\omega_0 t}\right) f(t-\tau)d\tau = e^{j\omega_0 t}\int_{-\infty}^{\infty} h(\tau)f(t-\tau)d\tau.$$

Hence,

$$y_f(t_0) = e^{j\omega_0 t_0}\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{j\omega t_0}d\omega.$$

On slides with title "Linear Systems Revisited" of Chapter 9, we have $S_{yy}(\omega) = |H_1(\omega - \omega_0)|^2 S_{xx}(\omega - \omega_0)$, which implies in this problem, $S_{y_v y_v}(\omega) = |H(\omega - \omega_0)|^2 S_{vv}(\omega - \omega_0)$. Hence,

$$\begin{aligned}
\gamma_o &= \frac{|y_f(t_0)|^2}{E[\boldsymbol{y}_v^2(t_0)]} = \frac{\left|\frac{1}{2\pi} e^{j\omega_0 t_0} \int_{-\infty}^{\infty} F(\omega) H(\omega) e^{j\omega t_0} d\omega\right|^2}{\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{vv}(\omega - \omega_0) |H(\omega - \omega_0)|^2 d\omega} \\
&= \frac{\left|\int_{-\infty}^{\infty} F(\omega) S_{vv}^{-1/2}(\omega) \cdot S_{vv}^{1/2}(\omega) H(\omega) e^{j\omega t_0} d\omega\right|^2}{2\pi \int_{-\infty}^{\infty} S_{vv}(\omega) |H(\omega)|^2 d\omega} \\
&\leq \frac{\int_{-\infty}^{\infty} \left|F(\omega) S_{vv}^{-1/2}(\omega)\right|^2 d\omega \cdot \int_{-\infty}^{\infty} \left|S_{vv}^{1/2}(\omega) H(\omega) e^{j\omega t_0}\right|^2 d\omega}{2\pi \int_{-\infty}^{\infty} S_{vv}(\omega) |H(\omega)|^2 d\omega} \quad \text{(Schwartz inequality)} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 S_{vv}^{-1}(\omega) d\omega,
\end{aligned}$$

with equality holds if, and only if, $k\left(F(\omega) S_{vv}^{-1/2}(\omega)\right)^* = S_{vv}^{-1/2}(\omega) H(\omega) e^{j\omega t_0}$ for some complex k, or equivalently $H(\omega) = k F^*(\omega) S_{vv}^{-1}(\omega) e^{-j\omega t_0}$. □

5. Find a non-causal (time-invariant) discrete filter $\mathtt{L}[\omega]$ such that the power spectrum of the discrete output process $\boldsymbol{x}[t]$ due to white discrete input $\boldsymbol{i}[t]$ with power spectrum density $S_{ii}[\omega] = 1$ is equal to

$$\frac{5 - 4\cos(\omega)}{10 - 6\cos(\omega)}.$$

(Hint: You shall also prove that the filter you design is non-causal.)

Solution. $S[z] = \frac{5 - 2(z + z^{-1})}{10 - 3(z + z^{-1})} = \frac{2(z - 1/2)(z - 2)}{3(z - 1/3)(z - 3)} = \frac{2(z - 1/2)}{3(z - 1/3)} \cdot \frac{2(z^{-1} - 1/2)}{3(z^{-1} - 1/3)}$

$$\begin{aligned}
&\Rightarrow \mathtt{L}[z] = \frac{2(z^{-1} - 1/2)}{3(z^{-1} - 1/3)} \quad \left(S[z] = \mathtt{L}[z]\mathtt{L}[1/z]\right) \\
&\Rightarrow \mathtt{L}[\omega] = \frac{2(e^{-j\omega} - 1/2)}{3(e^{-j\omega} - 1/3)} \\
&\Rightarrow |\mathtt{L}[\omega]|^2 = \left|\frac{2(e^{-j\omega} - 1/2)}{3(e^{-j\omega} - 1/3)}\right|^2 = \frac{4(e^{j\omega} - 1/2)(e^{-j\omega} - 1/2)}{9(e^{j\omega} - 1/3)(e^{-j\omega} - 1/3)} = \frac{5 - 4\cos(\omega)}{10 - 6\cos(\omega)}.
\end{aligned}$$

$$
\begin{aligned}
\mathtt{l}[\tau] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathtt{L}[\omega]e^{j\omega\tau}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2(e^{-j\omega}-1/2)}{3(e^{-j\omega}-1/3)}e^{j\omega\tau}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{2(e^{j\omega'}-1/2)}{3(e^{j\omega'}-1/3)}e^{j\omega'(-\tau)}d\omega', \ \omega' = -\omega \\
&= \cdots\cdots\cdots \\
&= \begin{cases} 0, & (-\tau) < 0 \\ 1-3^{-1}, & (-\tau) = 0 \\ -3^{-(1+(-\tau))}, & (-\tau) > 0 \end{cases} \\
&= \begin{cases} 0, & \tau > 0 \\ 1-3^{-1}, & \tau = 0 \\ -3^{-(1-\tau)}, & \tau < 0 \end{cases}
\end{aligned}
$$

Since the $\mathtt{l}[\tau] \neq 0$ for $\tau < 0$, it is a non-causal filter. □

6. (a) Prove that feeding a white process to the whitening filter of a discrete AR process $\boldsymbol{x}[t]$ will induce a discrete MA process.

   (b) Can a discrete AR process $\boldsymbol{x}[t]$ with line spectra be whitened by a linear time-invariant filter? Justify your answer.

Solution.

(a) By definition of an AR process, its whitening filter should be of the form

$$1/\mathtt{L}[z] = \frac{1 + a_1 z^{-1} + \cdots + a_n z^{-n}}{b_0} = \hat{b}_0 + \hat{b}_1 z^{-1} + \cdots + \hat{b}_n z^{-n},$$

where $\hat{b}_j = a_j/b_0$ for $0 \leq j \leq n$ and $a_0 = 1$. Then, the claim is valid by noting that this is exactly the form of the innovation filter of an MA process.

(b) Suppose the AR process $\boldsymbol{x}[t]$ with line spectra can be whitened by a linear time-invariant filter with transfer function $H[\omega]$. Then, the output power spectrum density $S_{yy}[\omega] = |H[\omega]|^2 S_{xx}[\omega]$ should be a (non-zero) constant for $-\pi \leq \omega < \pi$. However, for those ω not equal to the line spectra frequencies, $S_{yy}[\omega] = |H[\omega]|^2 S_{xx}[\omega] = 0$, a contradiction. Hence, the AR process with line spectra cannot be whitened by a linear time-invariant filter. □