Sample problems for the first quiz

1. Give a probability space

$$(S = \{\oplus, \ominus\}, \mathcal{F} = 2^S, P = \{p_1, p_2\} \text{ resp. for } S).$$

Note that p_1 is not necessarily equal to p_2.

(a) Construct a non-deterministic stationary random process $\{\boldsymbol{x}(t), t \in \Re\}$ that is defined over this probability space.
Hint: Stationarity implies that

$$P\{\zeta \in S : \boldsymbol{x}(t+c,\zeta) \in A\} = P\{\zeta \in S : \boldsymbol{x}(t,\zeta) \in A\}$$

for every $A \subset \Re$ and for every $t, c \in \Re$.

(b) Find the mean function and the autocorrelation function of $\boldsymbol{x}(t)$ that you define in (a).

(c) Verify that the autocorrelation function of $\boldsymbol{x}(t)$ is non-negative definite.

(d) Construct another non-deterministic stationary random process $\boldsymbol{y}(t)$ that is different from $\boldsymbol{x}(t)$ in (a).
Note: Here, "different" means $\boldsymbol{x}(t,\zeta) \neq \boldsymbol{y}(t,\zeta)$ for some $\zeta \in S$.

(e) Define a memoryless system $\{\boldsymbol{T}(x), x \in \mathcal{X}\}$ with system input $\boldsymbol{x}(t)$ in (a) and system output $\boldsymbol{y}(t)$ in (d), where $\boldsymbol{T}(x)$ is also defined over the same probability space.

(f) Is it possible to construct a (weakly) white stationary process $\boldsymbol{w}(t)$ over the given probability space? Justify your answer.
Note: A process $\boldsymbol{w}(t)$ is (weakly) *white* if $\boldsymbol{w}(t_1)$ and $\boldsymbol{w}(t_2)$ are *uncorrelated* for every $t_1 \neq t_2$.

Solution.

(a) For a stationary process, we shall have

$$P\{\zeta \in S : \boldsymbol{x}(t+c,\zeta) \in A\} = P\{\zeta \in S : \boldsymbol{x}(t,\zeta) \in A\}$$

for every $A \in \Re$ and for every $t, c \in \Re$. A non-deterministic process requires $\boldsymbol{x}(t,\oplus) \neq \boldsymbol{x}(t,\ominus)$. Thus, we may, for convenience, assign $\boldsymbol{x}(t,\oplus) = 1$ and $\boldsymbol{x}(t,\ominus) = -1$. This implies

$$\begin{aligned} P\{\zeta \in S : \boldsymbol{x}(t+c,\zeta) \in A\} &= P\{\zeta \in S : \boldsymbol{x}(t,\zeta) \in A\} \\ &= \begin{cases} P\{\ \} = 0, & \text{if } \pm 1 \notin A; \\ P\{\oplus\} = p_1, & \text{if } 1 \in A \text{ but } -1 \notin A; \\ P\{\ominus\} = p_2, & \text{if } -1 \in A \text{ but } 1 \notin A; \\ P\{\oplus,\ominus\} = 1, & \text{if } \pm 1 \in A. \end{cases} \end{aligned}$$

As a result (of taking $A = \{1\}$ and $A = \{-1\}$), we must have $\boldsymbol{x}(t+c, \oplus) = 1$ and $\boldsymbol{x}(t+c, \ominus) = -1$ for every $t, c \in \Re$.

(b)

$$\mu_x(t) = E[\boldsymbol{x}(t)] = P\{\oplus\} \cdot \boldsymbol{x}(t, \oplus) + P\{\ominus\} \cdot \boldsymbol{x}(t, \ominus) = p_1 - p_2$$

and

$$\begin{aligned} R_{xx}(t_1, t_2) &= E[\boldsymbol{x}(t_1)\boldsymbol{x}^*(t_2)] = P\{\oplus\} \cdot \boldsymbol{x}(t_1, \oplus)\boldsymbol{x}(t_2, \oplus) + P\{\ominus\} \cdot \boldsymbol{x}(t_1, \ominus)\boldsymbol{x}(t_2, \ominus) \\ &= p_1 + p_2 = 1. \end{aligned}$$

(c)

$$\sum_i \sum_j a_i a_j^* R_{xx}(t_i, t_j) = \sum_i \sum_j a_i a_j^* = \left(\sum_i a_i\right)\left(\sum_j a_j\right)^* = \left|\sum_i a_i\right|^2 \geq 0.$$

(d) Assign $\boldsymbol{y}(t, \oplus) = -1$ and $\boldsymbol{y}(t, \ominus) = 1$.

(e) Since $\boldsymbol{T}(x)$ is memoryless, it is nothing to do with time argument t. By $\boldsymbol{T}(\boldsymbol{x}(t,\zeta),\zeta) = \boldsymbol{y}(t,\zeta)$, we obtain

$$\boldsymbol{T}(\boldsymbol{x}(t, \oplus), \oplus) = \boldsymbol{y}(t, \oplus) \quad \text{and} \quad \boldsymbol{T}(\boldsymbol{x}(t, \ominus), \ominus) = \boldsymbol{y}(t, \ominus),$$

which implies

$$\boldsymbol{T}(x, \zeta) = \begin{cases} 1, & x = -1 \text{ and } \zeta = \ominus; \\ -1, & x = 1 \text{ and } \zeta = \oplus; \\ \text{arbitrary}, & x = -1 \text{ and } \zeta = \oplus; \\ \text{arbitrary}, & x = 1 \text{ and } \zeta = \ominus. \end{cases}$$

(f) Let $\boldsymbol{w}(t, \oplus) = a$ and $\boldsymbol{w}(t, \ominus) = b$. Then,

$$\mu_w(t) = E[\boldsymbol{w}(t)] = P\{\oplus\} \cdot \boldsymbol{x}(t, \oplus) + P\{\ominus\} \cdot \boldsymbol{x}(t, \ominus) = p_1 a + p_2 b,$$

which implies

$$\begin{aligned} E[\boldsymbol{w}(t_1)]E[\boldsymbol{w}^*(t_2)] &= (p_1 a + p_2 b)(p_1 a + p_2 b)^* \\ &= (p_1 a + p_2 b)(p_1 a^* + p_2 b^*) \\ &= p_1^2|a|^2 + 2p_1 p_2 \mathrm{Re}\{ab^*\} + p_2^2|b|^2. \end{aligned}$$

In addition,

$$\begin{aligned} R_{ww}(t_1, t_2) &= E[\boldsymbol{w}(t_1)\boldsymbol{w}^*(t_2)] \\ &= P\{\oplus\} \cdot \boldsymbol{w}^*(t_1, \oplus)\boldsymbol{w}(t_2, \oplus) + P\{\ominus\} \cdot \boldsymbol{w}(t_1, \ominus)\boldsymbol{w}^*(t_2, \ominus) \\ &= p_1|a|^2 + p_2|b|^2. \end{aligned}$$

Thus, we require

$$
\begin{aligned}
& p_1^2|a|^2 + 2p_1p_2\text{Re}\{ab^*\} + p_2^2|b|^2 = p_1|a|^2 + p_2|b|^2 \\
\iff\ & (p_1^2 - p_1)|a|^2 + 2p_1p_2\text{Re}\{ab^*\} + (p_2^2 - p_2)|b|^2 = 0 \\
\iff\ & (p_1^2 - p_1)|a|^2 - 2(p_1^2 - p_1)\text{Re}\{ab^*\} + (p_1^2 - p_1)|b|^2 = 0 \\
\iff\ & (p_1^2 - p_1)\big(|a|^2 - 2\text{Re}\{ab^*\} + |b|^2\big) = 0 \\
\iff\ & (p_1^2 - p_1)\big|a - b\big|^2 = 0
\end{aligned}
$$

which can be validated for any p_1 (only) under $a = b$. Consequently, $\boldsymbol{w}(t)$ is a (weakly) white stationary process defined over the given probability space if, and only if, it is a deterministic process.

2. (a) Prove that the autocorrelation function $R_{xx}(t_1, t_2)$ of a random process $\boldsymbol{x}(t)$ is non-negative definite, namely,

$$
\sum_i \sum_j a_i a_j^* R_{xx}(t_i, t_j) \geq 0 \quad \text{for any complex } a_i \text{ and } a_j. \tag{1}
$$

(b) If $\boldsymbol{x}(t)$ is WSS, determine the equivalent condition to (1) using $S_{xx}(\omega)$, where $S_{xx}(\omega)$ is the power spectrum density of $\boldsymbol{x}(t)$.

Solution.

(a)

$$
0 \leq E\left[\left|\sum_i a_i \boldsymbol{x}(t_i)\right|^2\right] = \sum_i \sum_j a_i a_j^* E[\boldsymbol{x}(t_i)\boldsymbol{x}^*(t_j)] = \sum_i \sum_j a_i a_j^* R_{xx}(t_i, t_j).
$$

(b)

$$
\begin{aligned}
& \sum_i \sum_j a_i a_j^* R_{xx}(t_i, t_j) \geq 0 \\
& \iff \sum_i \sum_j a_i a_j^* R_{xx}(t_i - t_j) \geq 0 \\
& \iff \sum_i \sum_j a_i a_j^* \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) e^{\imath\omega(t_i - t_j)} d\omega\right) \geq 0 \\
& \iff \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\left(\sum_i \sum_j a_i a_j^* e^{\imath\omega(t_i - t_j)}\right) d\omega \geq 0 \\
& \iff \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\left|\sum_i a_i e^{\imath\omega t_i}\right|^2 d\omega \geq 0
\end{aligned}
$$

3. A Poisson process $\boldsymbol{x}(t) \triangleq \boldsymbol{n}[0, t)$ is one that satisfies

   i) the number of Poisson point occurrences at $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3, \cdots\}$ in an interval $[t_1, t_2)$ is a Poisson random variable with parameter $\lambda(t_2 - t_1)$, i.e.,

$$\Pr\{\boldsymbol{n}[t_1, t_2) = k\} = \frac{e^{-\lambda(t_2-t_1)}\left[\lambda(t_2 - t_1)\right]^k}{k!},$$

   ii) and $\boldsymbol{n}[t_1, t_2)$ and $\boldsymbol{n}[t_3, t_4)$ are independent if $[t_1, t_2)$ and $[t_3, t_4)$ are non-overlapping intervals.

   Do the following problems.

   (a) Determine the mean and autocorrelation functions of $\boldsymbol{x}(t)$.

   (b) Determine $E[\boldsymbol{x}(t_1)\boldsymbol{x}(t_2)\boldsymbol{x}(t_3)]$ for $t_1 < t_2 < t_3$.
   Note: $E[\boldsymbol{n}^3[0, t)] = \lambda^3 t^3 + 3\lambda^2 t^2 + \lambda t$ and $E[\boldsymbol{n}^2[0, t)] = \lambda^2 t^2 + \lambda t$.

   Solution.

   (a) See Slides 9-48 and 9-49.

   (b) For $t_1 < t_2 < t_3$,

$$\begin{aligned}
E[\boldsymbol{x}(t_1)\boldsymbol{x}(t_2)\boldsymbol{x}(t_3)] &= E[\boldsymbol{n}[0,t_1)\cdot(\boldsymbol{n}[0,t_1)+\boldsymbol{n}[t_1,t_2))\cdot(\boldsymbol{n}[0,t_1)+\boldsymbol{n}[t_1,t_2)+\boldsymbol{n}[t_2,t_3))] \\
&= E[\boldsymbol{n}^3[0,t_1)] + E[\boldsymbol{n}^2[0,t_1)\boldsymbol{n}[t_1,t_2)] + E[\boldsymbol{n}^2[0,t_1)\boldsymbol{n}[t_2,t_3)] \\
&\quad + E[\boldsymbol{n}^2[0,t_1)\boldsymbol{n}[t_1,t_2)] + E[\boldsymbol{n}[0,t_1)\boldsymbol{n}^2[t_1,t_2)] + E[\boldsymbol{n}[0,t_1)\boldsymbol{n}[t_1,t_2)\boldsymbol{n}[t_2,t_3)] \\
&= E[\boldsymbol{n}^3[0,t_1)] + E[\boldsymbol{n}^2[0,t_1)]E[\boldsymbol{n}[t_1,t_2)] + E[\boldsymbol{n}^2[0,t_1)]E[\boldsymbol{n}[t_1,t_3)] \\
&\quad + E[\boldsymbol{n}[0,t_1)]E[\boldsymbol{n}^2[t_1,t_2)] + E[\boldsymbol{n}[0,t_1)]E[\boldsymbol{n}[t_1,t_2)]E[\boldsymbol{n}[t_2,t_3)] \\
&= \left(\lambda^3 t_1^3 + 3\lambda^2 t_1^2 + \lambda t_1\right) + \left(\lambda^2 t_1^2 + \lambda t_1\right)\left(\lambda t_2 - \lambda t_1\right) + \left(\lambda^2 t_1^2 + \lambda t_1\right)\left(\lambda t_3 - \lambda t_1\right) \\
&\quad + \lambda t_1\left(\lambda^2(t_2 - t_1)^2 + \lambda(t_2 - t_1)\right) + \lambda t_1\left(\lambda t_2 - \lambda t_1\right)\left(\lambda t_3 - \lambda t_2\right) \\
&= \lambda^3 t_1^3 + 3\lambda^2 t_1^2 + \lambda t_1 + \lambda^3 t_1^2 t_2 - \lambda^3 t_1^3 + \lambda^2 t_1 t_2 - \lambda^2 t_1^2 + \lambda^3 t_1^2 t_3 - \lambda^3 t_1^3 + \lambda^2 t_1 t_3 - \lambda^2 t_1^2 \\
&\quad + \lambda^3 t_1^3 + \lambda^3 t_1 t_2^2 - 2\lambda^3 t_1^2 t_2 + \lambda^2 t_1 t_2 - \lambda^2 t_1^2 + \lambda^3 t_1 t_2 t_3 - \lambda^3 t_1 t_2^2 - \lambda^3 t_1^2 t_3 + \lambda^3 t_1^2 t_2 \\
&= \lambda t_1 + \lambda^2(2t_1 t_2 + t_1 t_3) + \lambda^3 t_1 t_2 t_3 \\
&\quad \text{or } = \lambda^2 t_1 t_2 + \lambda t_1(1 + \lambda t_2)(1 + \lambda t_3).
\end{aligned}$$

4. (a) Prove that the process $\boldsymbol{x}(t) = \boldsymbol{a}\cos(\omega t) + \boldsymbol{b}\sin(\omega t)$ is WSS if, and only if, $\boldsymbol{a}$ and $\boldsymbol{b}$ are uncorrelated zero mean with equal variance.

   (b) Disprove the statement that if the process $\boldsymbol{x}(t) = a\cos(\boldsymbol{\omega} t + \boldsymbol{\varphi})$ is WSS, then $\boldsymbol{\varphi}$ is uniformly distributed over $[-\pi, \pi)$.

   (c) Disprove the statement that if the complex process $\boldsymbol{z}(t) = ae^{j(\boldsymbol{\omega} t + \boldsymbol{\varphi})}$ is WSS, then $\boldsymbol{\varphi}$ is uniformly distributed over $[-\pi, \pi)$.

   Solution.

(a) The "if" proof has been provided in my slides. It remains to prove the "only if" part.

WSS of $\boldsymbol{x}(t) = \boldsymbol{a}\cos(\omega t) + \boldsymbol{b}\sin(\omega t)$ implies that

$$E[\boldsymbol{x}(t)] = E[\boldsymbol{a}]\cos(\omega t) + E[\boldsymbol{b}]\sin(\omega t) = C \text{ for any } t$$

which in turn implies that $E[\boldsymbol{a}] = C$ at $t = 0$ and $E[\boldsymbol{a}] = -C$ at $t = \pi/\omega$. Hence, $E[\boldsymbol{a}] = C = 0$. As a result, we have $E[\boldsymbol{b}]\sin(\omega t) = 0$, which immediately gives $E[\boldsymbol{b}] = 0$.

Next, we derive

$$\begin{aligned} E[\boldsymbol{x}(t+\tau)\boldsymbol{x}(t)] &= E[\boldsymbol{a}^2]\cos(\omega t)\cos(\omega(t+\tau)) + E[\boldsymbol{b}^2]\sin(\omega t)\sin(\omega(t+\tau)) \\ &\quad + E[\boldsymbol{ab}]\sin(\omega(2t+\tau)) \\ &= E[\boldsymbol{a}^2]\frac{\cos(\omega\tau) + \cos(\omega(2t+\tau))}{2} + E[\boldsymbol{b}^2]\frac{\cos(\omega\tau) - \cos(\omega(2t+\tau))}{2} \\ &\quad + E[\boldsymbol{ab}]\sin(\omega(2t+\tau)) \\ &= \frac{E[\boldsymbol{a}^2] + E[\boldsymbol{b}^2]}{2}\cos(\omega\tau) + \frac{E[\boldsymbol{a}^2] - E[\boldsymbol{b}^2]}{2}\cos(\omega(2t+\tau)) \\ &\quad + E[\boldsymbol{ab}]\sin(\omega(2t+\tau)) \end{aligned}$$

which is only a function of τ. Hence, $E[\boldsymbol{a}^2] = E[\boldsymbol{b}^2]$ and $E[\boldsymbol{ab}] = 0$.

(b) To disprove the statement, it suffices to provide a counterexample. Suppose $\boldsymbol{\varphi} = 0$, $\pi/2$, π and $3\pi/2$ with equal probability and is independent of $\boldsymbol{\omega}$. Then,

$$\begin{aligned} E[\boldsymbol{x}(t)] &= aE[\cos(\boldsymbol{\omega}t + \boldsymbol{\varphi})] \\ &= a\frac{1}{4}E[\cos(\boldsymbol{\omega}t)] - a\frac{1}{4}E[\sin(\boldsymbol{\omega}t)] - a\frac{1}{4}E[\cos(\boldsymbol{\omega}t)] + a\frac{1}{4}E[\sin(\boldsymbol{\omega}t)] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} E[\boldsymbol{x}(t_1)\boldsymbol{x}(t_2)] &= E\{a^2\cos(\boldsymbol{\omega}t_1 + \boldsymbol{\varphi})\cos(\boldsymbol{\omega}t_2 + \boldsymbol{\varphi})\} \\ &= E\left\{a^2\frac{\cos[\boldsymbol{\omega}(t_1 - t_2)] + \cos[\boldsymbol{\omega}(t_1 + t_2) + 2\boldsymbol{\varphi}]}{2}\right\} = \frac{a^2}{2}E[\cos(\boldsymbol{\omega}(t_1 - t_2))]. \end{aligned}$$

(c) Again, a counterexample is provided. Suppose $\boldsymbol{\varphi} = 0$ and π with equal probability and is independent of $\boldsymbol{\omega}$. Then,

$$\begin{aligned} E[\boldsymbol{z}(t)] &= E[ae^{j(\boldsymbol{\omega}t + \boldsymbol{\varphi})}] \\ &= aE[e^{j\boldsymbol{\omega}t}]E[e^{j\boldsymbol{\varphi}}] = 0 \end{aligned}$$

and

$$\begin{aligned} E[\boldsymbol{z}(t)\boldsymbol{z}^*(t+\tau)] &= E[ae^{j(\boldsymbol{\omega}t + \boldsymbol{\varphi})}a^*e^{-j(\boldsymbol{\omega}(t+\tau) + \boldsymbol{\varphi})}] \\ &= |a|^2E[e^{j\boldsymbol{\omega}\tau}] \end{aligned}$$