# Chapter 10 Random Walks and Other Applications

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# 10-3 Modulation

- Starting at the 9th line on page 463, the textbook wrote

> We shall show that $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$ is WSS iff the processes $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are such that
>
> $$R_{aa}(\tau) = R_{bb}(\tau) \qquad R_{ab}(\tau) = -R_{ba}(\tau) \qquad (10\text{-}126).$$

The forward part is correct, but the converse may not be right!

**Lemma** The process $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$ is WSS if $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are zero-mean WSS with $R_{aa}(\tau) = R_{bb}(\tau)$ and $R_{ab}(\tau) = -R_{ba}(\tau)$.

**Proof:** If $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are zero-mean WSS with $R_{aa}(\tau) = R_{bb}(\tau)$ and $R_{ab}(\tau) = -R_{ba}(\tau)$, then $E[\boldsymbol{x}(t)] = 0$ and

$$\begin{aligned}
E[\boldsymbol{x}(t_1)\boldsymbol{x}(t_2)] &= E\{[\boldsymbol{a}(t_1)\cos(\omega_0 t_1) - \boldsymbol{b}(t_1)\sin(\omega_0 t_1)][\boldsymbol{a}(t_2)\cos(\omega_0 t_2) - \boldsymbol{b}(t_2)\sin(\omega_0 t_2)]\} \\
&= R_{aa}(t_1 - t_2)\cos(\omega_0 t_1)\cos(\omega_0 t_2) - R_{ab}(t_1 - t_2)\cos(\omega_0 t_1)\sin(\omega_0 t_2) \\
&\quad - R_{ba}(t_1 - t_2)\sin(\omega_0 t_1)\cos(\omega_0 t_2) + R_{bb}(t_1 - t_2)\sin(\omega_0 t_1)\sin(\omega_0 t_2) \\
&= R_{aa}(t_1 - t_2)\cos[\omega_0(t_1 - t_2)] + R_{ab}(t_1 - t_2)\sin[\omega_0(t_1 - t_2)],
\end{aligned}$$

which indicates the WSS of $\boldsymbol{x}(t)$. $\square$

# 10-3 Modulation



**Fallacy** If the process $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$ is WSS, then $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are such that

$$R_{aa}(\tau) = R_{bb}(\tau) \quad \text{and} \quad R_{ab}(\tau) = -R_{ba}(\tau).$$

**Counterexample:**

- $\boldsymbol{a}(t) = \sin(\omega_0 t)$, $\boldsymbol{b}(t) = \cos(\omega_0 t)$, and $\boldsymbol{x}(t) = 0$.
- $\boldsymbol{x}(t)$ is WSS, but

$$R_{aa}(t_1, t_2) = E[\boldsymbol{a}(t_1)\boldsymbol{a}(t_2)] = \sin(\omega_0 t_1)\sin(\omega_0 t_2)$$

and

$$R_{bb}(t_1, t_2) = E[\boldsymbol{b}(t_1)\boldsymbol{b}(t_2)] = \cos(\omega_0 t_1)\cos(\omega_0 t_2)$$

are not equal and are not functions of only $(t_1 - t_2)$. □

What will be the correct statement?

**Lemma** Suppose $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are zero-mean jointly WSS. Then, the process $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$ is WSS if, and only if, $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are such that

$$R_{aa}(\tau) = R_{bb}(\tau) \quad \text{and} \quad R_{ab}(\tau) = -R_{ba}(\tau).$$

- The blue-colored presumption is actually given at the first line of Section 10.3.

# 10-3 Modulation

**Proof:**

**1. Forward:** Have been proved in Slide 10-1.

**2. Converse:** If $\boldsymbol{x}(t)$ is WSS, then

$$\begin{aligned}
E[\boldsymbol{x}(t_1)\boldsymbol{x}(t_2)] &= R_{aa}(t_1,t_2)\cos(\omega_0 t_1)\cos(\omega_0 t_2) - R_{ab}(t_1,t_2)\cos(\omega_0 t_1)\sin(\omega_0 t_2) \\
&\quad - R_{ba}(t_1,t_2)\sin(\omega_0 t_1)\cos(\omega_0 t_2) + R_{bb}(t_1,t_2)\sin(\omega_0 t_1)\sin(\omega_0 t_2) \\
&= R_{aa}(t_1,t_2)\frac{\cos[\omega_0(t_1-t_2)]+\cos[\omega_0(t_1+t_2)]}{2} \\
&\quad - R_{ab}(t_1,t_2)\frac{\sin[\omega_0(t_1+t_2)]-\sin[\omega_0(t_1-t_2)]}{2} \\
&\quad - R_{ba}(t_1,t_2)\frac{\sin[\omega_0(t_1+t_2)]+\sin[\omega_0(t_1-t_2)]}{2} \\
&\quad + R_{bb}(t_1,t_2)\frac{\cos[\omega_0(t_1-t_2)]-\cos[\omega_0(t_1+t_2)]}{2} \\
&= \frac{1}{2}\cos[\omega_0(t_1+t_2)][R_{aa}(t_1-t_2)-R_{bb}(t_1-t_2)] \quad \text{(must be zero!)} \\
&\quad + \frac{1}{2}\cos[\omega_0(t_1-t_2)][R_{aa}(t_1-t_2)+R_{bb}(t_1-t_2)] \quad \text{(depends only on } (t_1-t_2)) \\
&\quad - \frac{1}{2}\sin[\omega_0(t_1+t_2)][R_{ab}(t_1-t_2)+R_{ba}(t_1-t_2)] \quad \text{(must be zero!)} \\
&\quad + \frac{1}{2}\sin[\omega_0(t_1-t_2)][R_{ab}(t_1-t_2)-R_{ba}(t_1-t_2)] \quad \text{(depends only on } (t_1-t_2))
\end{aligned}$$

imply that

$$R_{aa}(t_1 - t_2) = R_{bb}(t_1 - t_2) \quad \text{and} \quad R_{aa}(t_1 - t_2) + R_{bb}(t_1 - t_2) = 2R_{aa}(t_1 - t_2)$$
$$R_{ab}(t_1 - t_2) = -R_{ba}(t_1 - t_2) \quad \text{and} \quad R_{ab}(t_1 - t_2) - R_{ba}(t_1 - t_2) = 2R_{ab}(t_1 - t_2).$$

□

**Remarks**

- The above lemma yields that

$$\boxed{R_{xx}(\tau) = R_{aa}(\tau)\cos(\omega_0\tau) + R_{ab}(\tau)\sin(\omega_0\tau)}$$

  for zero-mean WSS $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ with $R_{aa}(\tau) = R_{bb}(\tau)$ and $R_{ab}(\tau) = -R_{ba}(\tau)$.

- Define $\boldsymbol{w}(t) = \boldsymbol{a}(t) + j\boldsymbol{b}(t)$. Then, it is easy to see that

$$\boxed{\boldsymbol{x}(t) = \text{Re}\{\boldsymbol{w}(t)e^{j\omega_0 t}\}}$$

# 10-3 Modulation



- We can define a dual function of $\boldsymbol{x}(t)$ as:

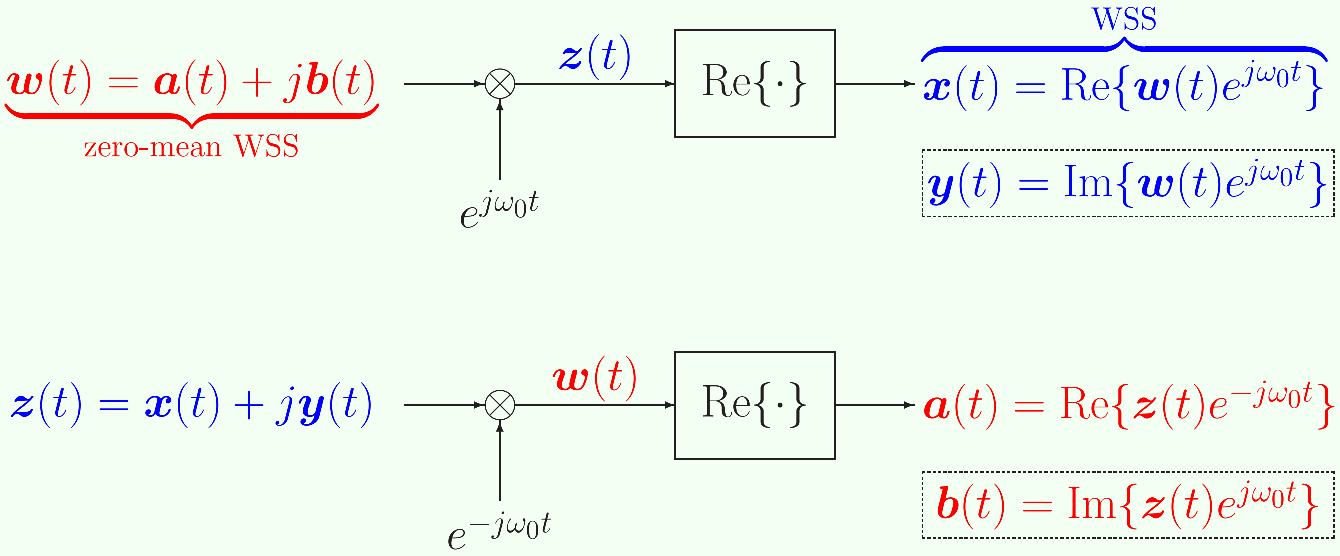
$$\boxed{\boldsymbol{y}(t) = \text{Im}\{\boldsymbol{w}(t)e^{j\omega_0 t}\} = \boldsymbol{a}(t)\sin(\omega_0 t) + \boldsymbol{b}(t)\cos(\omega_0 t)}$$

In summary,

$$\begin{aligned}
\boldsymbol{w}(t) &= \boldsymbol{a}(t) + j\boldsymbol{b}(t) \text{ with } R_{aa}(\tau) = R_{bb}(\tau), R_{ab}(\tau) = -R_{ba}(\tau) \\
\boldsymbol{x}(t) &= \text{Re}\{\boldsymbol{w}(t)e^{j\omega_0 t}\} \\
\boldsymbol{y}(t) &= \text{Im}\{\boldsymbol{w}(t)e^{j\omega_0 t}\} \\
\boldsymbol{z}(t) &= \boldsymbol{x}(t) + j\boldsymbol{y}(t) = \boldsymbol{w}(t)e^{j\omega_0 t}
\end{aligned}$$

In the sequel, we assume that $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are zero-mean jointly WSS and $\boldsymbol{x}(t)$ is WSS (so $R_{aa}(\tau) = R_{bb}(\tau)$ and $R_{ab}(\tau) = -R_{ba}(\tau)$).

**Graphical View**:



**Observation 1** $\boldsymbol{z}(t)$ is zero-mean WSS if $\boldsymbol{w}(t)$ is zero-mean WSS.

**Proof:** Observe that $\boldsymbol{z}(t) = \boldsymbol{h}(\tau;t) * \boldsymbol{w}(t)$ with $\boldsymbol{h}(\tau;t) = \boldsymbol{h}_1(\tau)\boldsymbol{h}_2(t)$, where $\boldsymbol{h}_1(\tau) = \delta(\tau)$ and $\boldsymbol{h}_2(t) = e^{j\omega_0 t}$. Hence, by Theorem 9-2 (cf. Slide 9-104),

$$\begin{aligned} R_{zz}(t+s,t) &= E\{\boldsymbol{h}_2(t+s)\boldsymbol{h}_2^*(t)[\boldsymbol{h}_1^*(-s) * \boldsymbol{h}_1(s) * R_{ww}(s)]\} \\ &= \boldsymbol{h}_2(t+s)\boldsymbol{h}_2^*(t) \cdot \delta(-s) * \delta(s) * R_{ww}(s) \\ &= e^{j\omega_0 s} R_{ww}(s). \end{aligned}$$

The proof is completed by noting that $E[\boldsymbol{z}(t)] = 0$. □

**Observation 2** $R_{xx}(\tau) = R_{yy}(\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$ if Observation 1 is true.

**Proof:** A direct consequence of the Lemma in Slide 10-2. Note that $\boldsymbol{a}(t) = \mathrm{Re}\{\boldsymbol{w}(t)\} = \mathrm{Re}\{\boldsymbol{z}(t)e^{-\omega_0 t}\} = \boldsymbol{x}(t)\cos(\omega_0 t) - \boldsymbol{y}(t)\sin(\omega_0 t)$ is WSS and $\boldsymbol{z}(t)$ is zero-mean WSS (equivalently, $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are zero-mean jointly WSS). □

**Observation 3** $R_{ww}(\tau) = 2R_{aa}(\tau) - 2jR_{ab}(\tau)$ and $R_{zz}(\tau) = 2R_{xx}(\tau) - 2jR_{xy}(\tau)$.

**Proof:** Follow Observation 2 and the definition of $\boldsymbol{w}(t)$ and $\boldsymbol{z}(t)$. □

# 10-3 Modulation

**Observation 4**

$$S_{xx}(\omega) = \frac{1}{4}[S_{zz}(\omega) + S_{zz}(-\omega)] = \frac{1}{4}[S_{ww}(\omega - \omega_0) + S_{ww}(-\omega - \omega_0)]$$

and

$$S_{xy}(\omega) = \frac{j}{4}[S_{zz}(\omega) - S_{zz}(-\omega)] = \frac{j}{4}[S_{ww}(\omega - \omega_0) - S_{ww}(-\omega - \omega_0)].$$

**Proof:** First, $R_{xx}(-\tau) = R_{xx}(\tau)$ implies $S_{xx}(-\omega) = S_{xx}(\omega)$.
Secondly, $R_{xy}(-\tau) = -R_{yx}(-\tau) = -R_{xy}(\tau)$ implies

$$\begin{aligned} S_{xy}(-\omega) &= \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j(-\omega)\tau} d\tau = \int_{-\infty}^{\infty} R_{xy}(-\tau) e^{-j\omega\tau} d\tau \\ &= -\int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau = -S_{xy}(\omega) \end{aligned}$$

Then, the observation follows from

$$\begin{aligned} S_{zz}(\omega) &= 2S_{xx}(\omega) - 2jS_{xy}(\omega) \quad \text{(Observation 3)} \\ S_{zz}(-\omega) &= 2S_{xx}(-\omega) - 2jS_{xy}(-\omega) = 2S_{xx}(\omega) + 2jS_{xy}(\omega) \\ S_{zz}(\omega) &= S_{ww}(\omega - \omega_0) \quad \text{(Observation 1)} \end{aligned}$$

□

**Rice's representation**

- The Lemma on Slide 10-1 states that the process

$$\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$$

is WSS if $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$ are zero-mean WSS with $R_{aa}(\tau) = R_{bb}(\tau)$ and $R_{ab}(\tau) = -R_{ba}(\tau)$.

- Rice claims that for any zero-mean WSS process $\boldsymbol{x}(t)$, there exists

$$\omega_0, \quad \boldsymbol{a}(t) \quad \text{and} \quad \boldsymbol{b}(t)$$

such that $\boldsymbol{x}(t)$ can be represented as $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$, which is named the *Rice's representation*. (Here, "=" in the MS sense.)

- Rice's representation is not *unique*!

$$\begin{aligned} \boldsymbol{a}(t) &= \mathrm{Re}\left\{(\boldsymbol{x}(t) + j\boldsymbol{y}(t))e^{-j\omega_0 t}\right\} \\ \boldsymbol{b}(t) &= \mathrm{Im}\left\{(\boldsymbol{x}(t) + j\boldsymbol{y}(t))e^{-j\omega_0 t}\right\}, \end{aligned}$$

for any $\omega_0$ and any zero-mean WSS $\boldsymbol{y}(t)$ satisfying $R_{xx}(\tau) = R_{yy}(\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$.

# Hilbert Transform



**How to choose $\boldsymbol{y}(t)$ that satisfies $R_{xx}(\tau) = R_{yy}(\tau)$ and $R_{xy}(\tau) = -R_{yx}(\tau)$**

- Choose or restrict $\boldsymbol{y}(t)$ to be $\boldsymbol{Y}(\omega) = \boldsymbol{X}(\omega)H(\omega)$.
- By Theorem 9-4 (cf. Slide 9-104),

$$S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega) \quad \text{and} \quad S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2.$$

- From $\boldsymbol{X}(\omega) = \boldsymbol{Y}(\omega)[1/H(\omega)]$ and Theorem 9-4 (exchanging the roles of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$), we obtain

$$S_{yx}(\omega) = S_{yy}(\omega)[1/H(\omega)]^* = S_{xx}(\omega)|H(\omega)|^2[1/H(\omega)]^* = S_{xx}(\omega)H(\omega).$$

- In order to have

$$S_{xx}(\omega) = S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 \quad \text{and} \quad S_{xy}(\omega) = -S_{yx}(\omega),$$

we require

$$\boxed{i)\ |H(\omega)|^2 = 1} \quad \text{and} \quad \boxed{ii)\ H(\omega) = -H^*(\omega)}.$$

# Hilbert Transform

- In addition, by $R_{xy}(-\tau) = -R_{yx}(-\tau) = -R_{xy}(\tau)$, we have $S_{xy}(-\omega) = -S_{xy}(\omega)$ or equivalently $S_{xx}(-\omega)H^*(-\omega) = -S_{xx}(\omega)H^*(\omega)$.

  Together with $S_{xx}(\omega) = S_{xx}(-\omega)$, we require $\boxed{iii)\ H(-\omega) = -H(\omega)}$.

- $\boxed{i)\ |H(\omega)|^2 = 1}$ implies $H(\omega) = e^{j\phi(\omega)}$ for some $\phi(\omega)$.

- $e^{j\phi(\omega)} = \underbrace{\boxed{H(\omega) = -H^*(\omega)}}_{ii)} = -e^{-j\phi(\omega)}$ implies $e^{j2\phi(\omega)} = -1$, which in turns implies

$$\phi(\omega) = \left(k(\omega) + \frac{1}{2}\right)\pi$$

  for some integer function $k(\omega)$. For convenience, we restrict $k(\omega) \in \{0, 1\}$.

- Hence,

$$H(\omega) = je^{j\pi k(\omega)} = j(-1)^{k(\omega)}.$$

- Finally, $\boxed{iii)\ H(-\omega) = -H(\omega)}$ implies $k(\omega) \neq k(-\omega)$ for $k(\omega) \in \{0, 1\}$.

**Claim** For a given $S_{xx}(\omega)$, the choice of Hilbert transform $\boldsymbol{y}(t)$ of $\boldsymbol{x}(t)$ minimizes the average rate of variation of the complex envelope of $\boldsymbol{x}(t)$, namely, $E[|\boldsymbol{w}'(t)|^2]$.

**Proof:**

- Since the transfer function of a differentiator is $j\omega$,

$$S_{w'w'}(\omega) = S_{ww}(\omega)|j\omega|^2 = S_{ww}(\omega)\omega^2.$$

- Observation 1 indicates that $S_{ww}(\omega) = S_{zz}(\omega + \omega_0)$.
- Hence, the problem becomes to minimize

$$\begin{aligned} M &\triangleq 2\pi E[|\boldsymbol{w}'(t)|^2] = \int_{-\infty}^{\infty} S_{w'w'}(\omega)d\omega \\ &= \int_{-\infty}^{\infty} \omega^2 S_{ww}(\omega)d\omega = \int_{-\infty}^{\infty} (\omega - \omega_0)^2 S_{zz}(\omega)d\omega. \end{aligned} \quad (10.1)$$

- For a selected $S_{zz}(\omega)$, the best $\bar{\omega}_0$ that minimizes (10.1) should satisfy

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{\infty} \omega S_{zz}(\omega)d\omega}{\int_{-\infty}^{\infty} S_{zz}(\omega)d\omega}.$$

- Taking $\bar{\omega}_0$ into (10.1) yields

$$M = \int_{-\infty}^{\infty} (\omega^2 - \bar{\omega}_0^2) S_{zz}(\omega) d\omega.$$

Observe that

$$\begin{aligned} M &= \int_{-\infty}^{\infty} (\omega^2 - \bar{\omega}_0^2) S_{zz}(-\omega) d\omega \\ &= \frac{1}{2}\left( \int_{-\infty}^{\infty} (\omega^2 - \bar{\omega}_0^2) S_{zz}(\omega) d\omega + \int_{-\infty}^{\infty} (\omega^2 - \bar{\omega}_0^2) S_{zz}(-\omega) d\omega \right) \\ &= \frac{1}{2}\left( \int_{-\infty}^{\infty} (\omega^2 - \bar{\omega}_0^2) [S_{zz}(\omega) + S_{zz}(-\omega)] d\omega \right) \\ &= 2\int_{-\infty}^{\infty} (\omega^2 - \bar{\omega}_0^2) S_{xx}(\omega) d\omega, \quad \left( = 2\int_{-\infty}^{\infty} \omega^2 S_{xx}(\omega) d\omega - 2\bar{\omega}_0^2 \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega \right) \end{aligned}$$

where the last equality follows from $4S_{xx}(\omega) = S_{zz}(\omega) + S_{zz}(-\omega)$ (cf. Observation 4). As a result, it suffices to maximize $\bar{\omega}_0^2$ for the minimization of $M$ for a given $S_{xx}(\omega)$.

- By $S_{zz}(\omega) = 2S_{xx}(\omega) - 2jS_{xy}(\omega)$ (Observation 3), $S_{xx}(\omega) = S_{xx}(-\omega)$ (Observation 4) and $R_{xy}(-\tau) = -R_{yx}(-\tau) = -R_{xy}(\tau)$ (or equivalently, $S_{xy}(-\omega) = -S_{xy}(\omega)$), we have

$$\int_{-\infty}^{\infty} \omega S_{zz}(\omega) d\omega \;=\; 2\int_{-\infty}^{\infty} (-j)\omega S_{xy}(\omega) d\omega = 4\int_{0}^{\infty} (-j)\omega S_{xy}(\omega) d\omega.$$

Also,

$$\begin{aligned}\int_{-\infty}^{\infty} S_{zz}(\omega) d\omega \;&=\; \int_{-\infty}^{\infty} S_{zz}(-\omega) d\omega = \frac{1}{2}\left(\int_{-\infty}^{\infty} S_{zz}(\omega) d\omega + \int_{-\infty}^{\infty} S_{zz}(-\omega) d\omega\right) \\ &=\; 2\int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = 4\int_{0}^{\infty} S_{xx}(\omega) d\omega.\end{aligned}$$

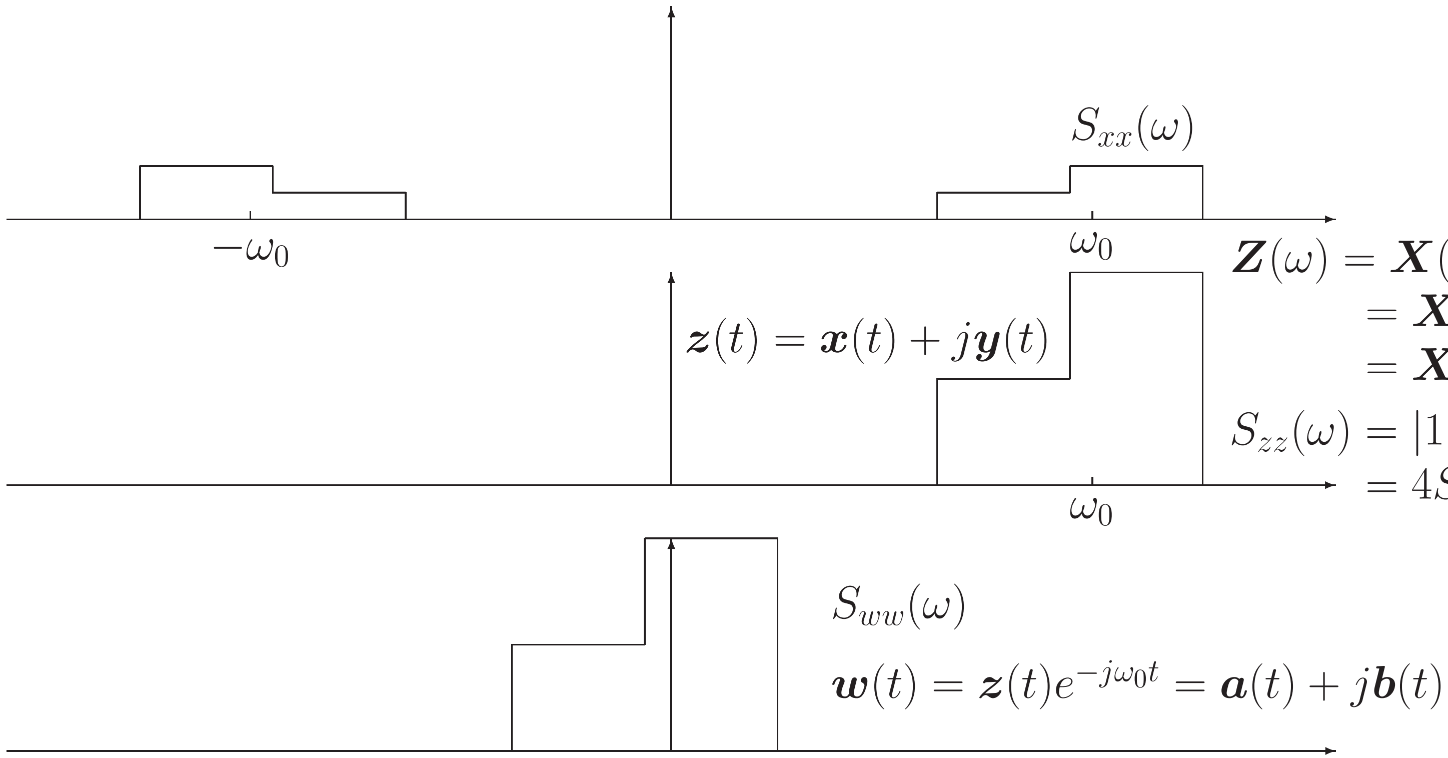
Hence,

$$\bar{\omega}_0 = \frac{\int_0^\infty (-j)\omega S_{xy}(\omega) d\omega}{\int_0^\infty S_{xx}(\omega) d\omega} = \frac{\int_0^\infty (-j)\omega S_{xx}(\omega) H^*(\omega) d\omega}{\int_0^\infty S_{xx}(\omega) d\omega} = \frac{\int_0^\infty \omega S_{xx}(\omega)(-1)^{k(\omega)+1} d\omega}{\int_0^\infty S_{xx}(\omega) d\omega}.$$

Consequently, the maximum $\bar{\omega}_0$ is obtained if $(-1)^{k(\omega)+1} = 1$ for $\omega > 0$. $\square$

## Hilbert transform $\boldsymbol{y}(t)$ of $\boldsymbol{x}(t)$

- Since $k(\omega) = 0$ for $\omega < 0$ and $k(\omega) = 1$ for $\omega > 0$, $H(\omega) = j(-1)^{k(\omega)} = -j\text{sgn}(\omega)$ is the Hilbert transformer, and $\omega_0 = \int_0^\infty wS_{xx}(\omega)d\omega / \int_0^\infty S_{xx}(\omega)d\omega$.




$$
\begin{aligned}
\boldsymbol{Z}(\omega) &= \boldsymbol{X}(\omega) + j\boldsymbol{Y}(\omega) \\
&= \boldsymbol{X}(\omega)[1 + jH(\omega)] \\
&= \boldsymbol{X}(\omega)[1 + \text{sgn}(\omega)] \\
S_{zz}(\omega) &= |1 + \text{sgn}(\omega)|^2 S_{xx}(\omega) \\
&= 4S_{xx}(\omega)\mathbf{1}\{\omega > 0\}
\end{aligned}
$$

# Terminologies

$$
\begin{aligned}
\boldsymbol{w}(t) &= \boldsymbol{a}(t) + j\boldsymbol{b}(t) && \text{Complex envelope or Lowpass signal} \\
&= \boldsymbol{r}(t)e^{j\boldsymbol{\varphi}(t)} \\
\boldsymbol{x}(t) &= \mathrm{Re}\{\boldsymbol{w}(t)e^{j\omega_0 t}\} && \text{Bandpass signal} \\
&= \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t) \\
&= \boldsymbol{r}(t)\cos[\omega_0 t + \boldsymbol{\varphi}(t)] \\
\boldsymbol{a}(t) & && \text{Inphase component} \\
\boldsymbol{b}(t) & && \text{Quadrature component} \\
\boldsymbol{\omega}_i(t) &= \frac{\partial}{\partial t}(\omega_0 t + \boldsymbol{\varphi}(t)) = \omega_0 + \boldsymbol{\varphi}'(t) && \text{Instantaneous freququency}
\end{aligned}
$$

**Definition (Bandpass)** A process $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$ is called *bandpass* if $S_{xx}(\omega) = 0$ for $|\omega|$ outside an interval $(\omega_1, \omega_2)$.

**Definition (Narrowband)** A bandpass process $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$ is called *narrowband* or *quasimonochromatic* if $|\omega_2 - \omega_1| \ll \omega_0$.

**Definition (Monochromatic)** A bandpass process $\boldsymbol{x}(t) = \boldsymbol{a}(t)\cos(\omega_0 t) - \boldsymbol{b}(t)\sin(\omega_0 t)$ is called *monochromatic* if $S_{xx}(\omega)$ is an impulse.

# Instantaneous Frequency and Optimal Center Frequency 

The optimal center frequency is given by

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{\infty} \omega S_{zz}(\omega) d\omega}{\int_{-\infty}^{\infty} S_{zz}(\omega) d\omega} = j \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\omega) S_{zz}(\omega) d\omega}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{zz}(\omega) d\omega} = j \frac{E[\boldsymbol{z}(t)(\boldsymbol{z}'(t))^*]}{E[|\boldsymbol{z}(t)|^2]}.$$

Now observe that $\boldsymbol{z}(t) = \boldsymbol{r}(t)e^{j(\omega_0 t + \boldsymbol{\varphi}(t))}$ implies $\boldsymbol{z}'(t) = [\boldsymbol{r}'(t) + j\boldsymbol{r}(t)\boldsymbol{\omega}_i(t)]e^{j(\omega_0 t + \boldsymbol{\varphi}(t))}$. Then,

$$\begin{aligned} E[\boldsymbol{z}(t)(\boldsymbol{z}'(t))^*] &= E\left[\boldsymbol{r}(t)\cancel{e^{j(\omega_0 t + \boldsymbol{\varphi}(t))}}[\boldsymbol{r}'(t) - j\boldsymbol{r}(t)\boldsymbol{\omega}_i(t)]\cancel{e^{-j(\omega_0 t + \boldsymbol{\varphi}(t))}}\right] \\ &= E\left[\boldsymbol{r}(t)\boldsymbol{r}'(t)\right] - jE\left[\boldsymbol{r}^2(t)\boldsymbol{\omega}_i(t)\right]. \end{aligned}$$

Since $S_{xx}(\omega) = S_{xx}(-\omega) = S_{yy}(\omega) = S_{yy}(-\omega)$, we have

$$E[\boldsymbol{x}(t)\boldsymbol{x}'(t)] = E[\boldsymbol{y}(t)\boldsymbol{y}'(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-j\omega) S_{xx}(\omega) d\omega = 0,$$

and $E[\boldsymbol{r}(t)\boldsymbol{r}'(t)] = E[\boldsymbol{x}(t)\boldsymbol{x}'(t)] + E[\boldsymbol{y}(t)\boldsymbol{y}'(t)] = 0$.
This concludes that $E[\boldsymbol{z}(t)(\boldsymbol{z}'(t))^*] = -jE\left[\boldsymbol{r}^2(t)\boldsymbol{\omega}_i(t)\right]$, which together with $E[|\boldsymbol{z}(t)|^2] = E[|\boldsymbol{r}(t)|^2]$ implies

$$\boxed{\text{optimal carrier freq } \bar{\omega}_0 = \frac{E[\boldsymbol{r}^2(t)\boldsymbol{\omega}_i(t)]}{E[\boldsymbol{r}^2(t)]} = \text{weighted average of } \boldsymbol{\omega}_i \left( = \omega_0 + \frac{E[\boldsymbol{r}^2(t)\boldsymbol{\varphi}'(t)]}{E[\boldsymbol{r}^2(t)]}\right)}$$

# Frequency Modulation



Consider the *frequency modulation with modulation index* $\lambda$.

Let $\boldsymbol{x}(t) = \cos[\omega_0 t + \lambda \boldsymbol{\varphi}(t) + \boldsymbol{\varphi}_0]$, where $\boldsymbol{\varphi}(t) = \int_0^t \boldsymbol{c}(\alpha) d\alpha$.

$$\begin{aligned} \boldsymbol{r}(t) &= 1 \\ \boldsymbol{w}(t) &= e^{j(\lambda\boldsymbol{\varphi}(t)+\boldsymbol{\varphi}_0)} \\ \boldsymbol{x}(t) &= \mathrm{Re}\{\boldsymbol{w}(t)e^{j\omega_0 t}\} \\ \boldsymbol{z}(t) &= \boldsymbol{w}(t)e^{j\omega_0 t} \end{aligned}$$

**Theorem 10-3** If $\boldsymbol{c}(t)$ is SSS, and $\boldsymbol{\varphi}_0 \perp\!\!\!\perp \boldsymbol{c}(t)$, and $E[e^{j\boldsymbol{\varphi}_0}] = E[e^{j2\boldsymbol{\varphi}_0}] = 0$, then $\boldsymbol{x}(t)$ is zero-mean WSS, where "$\perp\!\!\!\perp$" means "independent."

**Proof:** Since $E[\boldsymbol{z}(t)] = E[\boldsymbol{w}(t)e^{j\omega_0 t}] = E[e^{j\lambda\boldsymbol{\varphi}(t)}]E[e^{j\boldsymbol{\varphi}_0}]e^{j\omega_0 t} = 0$,

$$E[\boldsymbol{x}(t)] = E\left[\mathrm{Re}\left\{\boldsymbol{z}(t)\right\}\right] = \mathrm{Re}\left\{E\left[\boldsymbol{z}(t)\right]\right\} = 0.$$

In addition,

$$\begin{aligned} R_{xx}(t+\tau, t) &= E[\boldsymbol{x}(t+\tau)\boldsymbol{x}(t)] \quad \text{(because } \boldsymbol{x}(t) \text{ real)} \\ &= \frac{1}{4}E\left[(\boldsymbol{z}(t+\tau) + \boldsymbol{z}^*(t+\tau))(\boldsymbol{z}(t) + \boldsymbol{z}^*(t))\right]. \quad \left(\text{since } \boldsymbol{x}(t) = \frac{1}{2}(\boldsymbol{z}(t) + \boldsymbol{z}^*(t))\right) \end{aligned}$$

# Frequency Modulation



Observe that

$$\begin{aligned} E[\boldsymbol{z}(t+\tau)\boldsymbol{z}(t)] &= E\left[e^{j(\omega_0(t+\tau)+\lambda\boldsymbol{\varphi}(t+\tau)+\boldsymbol{\varphi}_0)}e^{j(\omega_0 t+\lambda\boldsymbol{\varphi}(t)+\boldsymbol{\varphi}_0)}\right] \\ &= E\left[e^{j(\omega_0(2t+\tau)+\lambda\boldsymbol{\varphi}(t+\tau)+\lambda\boldsymbol{\varphi}(t))}\right]E\left[e^{j2\boldsymbol{\varphi}_0}\right] = 0, \end{aligned}$$

and

$$\begin{aligned} R_{zz}(t+\tau,t) &= E[\boldsymbol{z}(t+\tau)\boldsymbol{z}^*(t)] \\ &= E\left[e^{j(\omega_0(t+\tau)+\lambda\boldsymbol{\varphi}(t+\tau)+\boldsymbol{\varphi}_0)}e^{-j(\omega_0 t+\lambda\boldsymbol{\varphi}(t)+\boldsymbol{\varphi}_0)}\right] \\ &= e^{j\omega_0\tau}E\left[e^{j\lambda[\boldsymbol{\varphi}(t+\tau)-\boldsymbol{\varphi}(t)]}\right] \\ &= e^{j\omega_0\tau}E\left[e^{j\lambda\int_t^{t+\tau}\boldsymbol{c}(\alpha)d\alpha}\right] \\ &= e^{j\omega_0\tau}E\left[e^{j\lambda\int_0^{\tau}\boldsymbol{c}(\alpha)d\alpha}\right] \quad \text{(because } \boldsymbol{c}(t) \text{ is SSS)} \\ &= e^{j\omega_0\tau}E[e^{j\lambda\boldsymbol{\varphi}(\tau)}]. \end{aligned}$$

Consequently,

$$R_{xx}(t+\tau,t) = \frac{1}{4}\left(R_{zz}(\tau)+R_{zz}^*(\tau)\right),$$

which together with $E[\boldsymbol{x}(t)] = 0$ implies the WSS of $\boldsymbol{x}(t)$. □

**Remarks on Theorem 10-3**

- From the proof of Theorem 10-3, we also learn that:

$$R_{xx}(\tau) = \frac{1}{2}\text{Re}\{R_{zz}(\tau)\} \quad \text{and} \quad R_{ww}(\tau) = E[e^{j\lambda\boldsymbol{\varphi}(\tau)}] \text{ (since } R_{zz}(\tau) = R_{ww}(\tau)e^{j\omega_0\tau})$$

- In addition, $\boldsymbol{x}(t)$ is in general not WSS if $\varphi_0$ is deterministic since $E[e^{j\boldsymbol{\varphi}_0}] = e^{j\varphi_0} \neq 0$.

- Further classification of $\boldsymbol{x}(t)$:

  – The process $\boldsymbol{x}(t)$ is generally classified to "*phase modulated*" if the statistics of $\boldsymbol{\varphi}(t)$ is known (i.e., $\boldsymbol{\varphi}(t)$ is the information process).

    $R_{xx}(\tau) = \frac{1}{2}\text{Re}\left\{E\left[e^{j\lambda\boldsymbol{\varphi}(\tau)}\right]e^{j\omega_0\tau}\right\}$ is well-defined for the random process $\boldsymbol{\varphi}(t)$ because "any finite-dimensional (including one-dimensional) sample distribution is well-defined for a random process."

  – The process $\boldsymbol{x}(t)$ is generally classified to "*frequency modulated*" if the statistics of $\boldsymbol{c}(t)$ is known (i.e., $\boldsymbol{c}(t)$ is the information process).

    $R_{xx}(\tau) = \frac{1}{2}\text{Re}\left\{E\left[e^{j\lambda\boldsymbol{\varphi}(\tau)}\right]e^{j\omega_0\tau}\right\}$ may not be well-defined even if the distribution of $\boldsymbol{c}(t)$ is known. An extreme example is that $\boldsymbol{c}(t)$ is not Lebesque-integrable in $t$.

## Woodward's Theorem

**Remarks on *frequency modulation***

- In order for $\boldsymbol{x}(t)$ to be zero-mean WSS, Theorem 10-3 (cf. Slide 10-18) requires that $\boldsymbol{c}(t)$ is SSS, and $\boldsymbol{\varphi}_0 \perp\!\!\!\perp \boldsymbol{c}(t)$, and $E[e^{j\boldsymbol{\varphi}_0}] = E[e^{j2\boldsymbol{\varphi}_0}] = 0$.
- Without SSS of $\boldsymbol{c}(t)$, $\boldsymbol{x}(t)$ may not be WSS, and the calculation of $S_{xx}(\omega)$ (or $R_{xx}(\tau)$) lacks of its footing!
- **Question** is that how to approximate $S_{xx}(\omega)$ under known statistics of SSS $\boldsymbol{c}(t)$?

  **Answer**: Woodward's Theorem.

# Woodward's Theorem



**Theorem 10-4 (Woodward's Theorem)** If the process $\boldsymbol{c}(t)$ is continuous and SSS with marginal density $f_{\boldsymbol{c}}(c)$, and also if $\boldsymbol{c}(t) \perp\!\!\!\perp \boldsymbol{\varphi}_0$, and $E[e^{j\boldsymbol{\varphi}_0}] = E[e^{j2\boldsymbol{\varphi}_0}] = 0$, then for large $\lambda$,

$$S_{xx}(\omega) \approx \frac{\pi}{2\lambda}\left[f_{\boldsymbol{c}}\left(\frac{\omega-\omega_0}{\lambda}\right) + f_{\boldsymbol{c}}\left(\frac{-\omega-\omega_0}{\lambda}\right)\right].$$

**Proof:**

- By the continuity of $\boldsymbol{c}(t)$,

$$\boldsymbol{\varphi}(t) \approx \boldsymbol{c}(0)t \text{ for } |t| < \tau_0$$

for some $\tau_0$ sufficiently small. So,

$$\boxed{\boldsymbol{x}(t) = \text{Re}\{\boldsymbol{z}(t)\}} \approx \boxed{\bar{\boldsymbol{x}}(t) \triangleq \text{Re}\{\bar{\boldsymbol{z}}(t)\}} \text{ for } |t| < \tau_0,$$

where

$$\boldsymbol{z}(t) \triangleq e^{j(\omega_0 t + \lambda\int_0^t \boldsymbol{c}(\alpha)d\alpha + \boldsymbol{\varphi}_0)} \quad \text{and} \quad \bar{\boldsymbol{z}}(t) \triangleq e^{j(\omega_0 t + \lambda t \boldsymbol{c}(0) + \boldsymbol{\varphi}_0)}.$$

Take a look at the lemmas on Slide 9-105 and compare them with $\bar{\boldsymbol{z}}(t)$!

# Woodward's Theorem



- Treating $t\boldsymbol{c}(0)$ as $\int_0^t \bar{\boldsymbol{c}}(\alpha)d\alpha$ with $\bar{\boldsymbol{c}}(t) = \boldsymbol{c}(0)$ for every $t \in \Re$, we observe that $\bar{\boldsymbol{c}}(t)$ is SSS and $\bar{\boldsymbol{c}}(t) \perp\!\!\!\perp \boldsymbol{\varphi}_0$, and hence, we can follow the proof of Theorem 10-3 to obtain:

$$R_{\bar{z}\bar{z}}(\tau) \;=\; E\left[e^{j\lambda\tau\boldsymbol{c}(0)}\right]e^{j\omega_0\tau} = e^{j\omega_0\tau}\int_{-\infty}^{\infty} f_{\boldsymbol{c}}(c)e^{j\lambda\tau c}dc = \frac{1}{\lambda}e^{j\omega_0\tau}\int_{-\infty}^{\infty} f_{\boldsymbol{c}}\left(\frac{u}{\lambda}\right)e^{ju\tau}du,$$

which implies

$$\begin{aligned}
S_{\bar{z}\bar{z}}(\omega) &= \int_{-\infty}^{\infty} R_{\bar{z}\bar{z}}(\tau)e^{-j\omega\tau}d\tau \\
&= \frac{1}{\lambda}\int_{-\infty}^{\infty} f_{\boldsymbol{c}}\left(\frac{u}{\lambda}\right)\int_{-\infty}^{\infty} e^{-j\tau(\omega-\omega_0-u)}d\tau du \\
&= \frac{1}{\lambda}\int_{-\infty}^{\infty} f_{\boldsymbol{c}}\left(\frac{u}{\lambda}\right)\cdot 2\pi\delta(\omega-\omega_0-u)du \\
&= \frac{2\pi}{\lambda}f_{\boldsymbol{c}}\left(\frac{\omega-\omega_0}{\lambda}\right).
\end{aligned}$$

Then, Observation 4 (cf. Slide 10-8) implies that

$$S_{\bar{x}\bar{x}}(\omega) = \frac{1}{4}\left[S_{\bar{z}\bar{z}}(\omega) + S_{\bar{z}\bar{z}}(-\omega)\right] = \frac{\pi}{2\lambda}\left[f_{\boldsymbol{c}}\left(\frac{\omega-\omega_0}{\lambda}\right) + f_{\boldsymbol{c}}\left(\frac{-\omega-\omega_0}{\lambda}\right)\right].$$

# Woodward's Theorem



- Now in order for $S_{\bar{x}\bar{x}}(\omega)$ to well-approximate $S_{xx}(\omega)$, we hope that

$$R_{\bar{z}\bar{z}}(\tau) = e^{j\omega_0\tau}E[e^{j\lambda\tau\boldsymbol{c}(0)}] \tag{10.2}$$

well-approximates

$$R_{zz}(\tau) = e^{j\omega_0\tau}E\left[e^{j\lambda\int_0^\tau \boldsymbol{c}(\alpha)d\alpha}\right] \tag{10.3}$$

for most $\tau \in \Re$. We already know that (10.2) is close to (10.3) for $|\tau| < \tau_0$. As for $|\tau| \geq \tau_0$, because $\boldsymbol{c}(0)$ and $\int_0^\tau \boldsymbol{c}(\alpha)d\alpha$ assume to have densities, we can make:

$$E[e^{j\lambda\tau\boldsymbol{c}(0)}] \approx 0 \quad \text{and} \quad E\left[e^{j\lambda\int_0^\tau \boldsymbol{c}(\alpha)d\alpha}\right] \approx 0 \quad \text{if } \lambda \text{ is sufficiently large.}$$

This proves the requirement that "for large $\lambda$" in the theorem. $\square$

We will see how well the approximate in Woodward's theorem is for the special case that $\boldsymbol{c}(t)$ is a Gaussian process.

**Riemann-Lebesgue Theorem** (Thm. 26.1 in P. Billingsley, *Probability and Measure*, 3rd Ed., Wiley, 1995) If $X$ has a density, then $\varphi_X(t) \triangleq E[e^{jtX}] \stackrel{|t|\to\infty}{\longrightarrow} 0$.

# Gaussian Processes

**Definition (Gaussian process)** [p. 122, *Random Processes: A Mathematical Approach for Engineers*, R. M. Gray & L. D. Davisson] A random process $\{\boldsymbol{x}(t), t \in \mathcal{I}\}$ is said to be a *Gaussian random process* if all finite collections of samples of the process are Gaussian random vectors.

- This is exactly the definition used in the textbook (cf. Slide 9-42).

**Lemma** [p. 122, *Random Processes: A Mathematical Approach for Engineers*, R. M. Gray & L. D. Davisson] A Gaussian random process is completely determined by a real-valued mean function $\mu(t)$ and a symmetric positive definite function $C(t_1, t_2)$.

- This is exactly what states in **Existence Theorem** in Slide 9-42.

**Definition (Gaussian process)** [p. 54, *Communication Systems*, 4th edition, S. Haykin] A random process $\boldsymbol{x}(t)$ is said to be a *Gaussian process*, if every (Lebesque-integrable) linear functional of $\boldsymbol{x}(t)$ in the form of

$$\boldsymbol{y} = \int_{-\infty}^{\infty} g(t)\boldsymbol{x}(t)dt$$

is a Gaussian random variable, provided that $\boldsymbol{y}$ has finite variance.

# Gaussian Processes



- By this definition, $\boldsymbol{\varphi}(t) = \int_0^t \boldsymbol{c}(\alpha)d\alpha$ is certainly a Gaussian process, if $\boldsymbol{c}(t)$ is a Gaussian process.

$$\begin{aligned}\int_{-\infty}^{\infty} g(t)\boldsymbol{\varphi}(t)dt &= \int_0^{\infty}\int_0^t g(t)\boldsymbol{c}(\alpha)d\alpha dt - \int_0^{\infty}\int_{-t}^0 g(-t)\boldsymbol{c}(\alpha)d\alpha dt \\ &= \int_0^{\infty}\boldsymbol{c}(\alpha)\int_{\alpha}^{\infty} g(t)dtd\alpha - \int_{-\infty}^{0}\boldsymbol{c}(\alpha)\int_{-\alpha}^{\infty} g(-t)dtd\alpha \\ &= \int_{-\infty}^{\infty}\tilde{g}(\alpha)\boldsymbol{c}(\alpha)d\alpha\end{aligned}$$

where $\tilde{g}(\alpha) \triangleq \mathbf{1}\{\alpha > 0\} \cdot \int_{\alpha}^{\infty} g(t)dt - \mathbf{1}\{\alpha < 0\} \cdot \int_{-\alpha}^{\infty} g(-t)dt$.

- Hakin's definition of Gaussian processes implies the definition in the textbook.

A random vector is Gaussian if every linear combination of the vector component is a Gaussian random variable.

- The converse is also true since every Lebesque-integrable function can be approximated by some Riemann-integrable function (cf. Slide 26-16 in my course: *Advanced Probability in Communications*). E.g., the integral of the Lebesque-integrable-but-Riemann-nonintegrable function that $f(x) = 0$ if $x$ is irrational, and 1, if $x$ is rational, can be approximated by the integral of the Riemann-integrable function $\bar{f}(x) = 0$. Thus, the integral result $\boldsymbol{y}$ can be obtained by taking finite number of samples of $g(t)\boldsymbol{x}(t)$, and then letting the number of samples go to infinity. The limiting distribution is certainly Gaussian because for each sampled number, the samples are constituted of a Gaussian vector by the text's definition.

# Examination of Woodward's Approximate



The accuracy of Woodward's approximate is determined by how well

$$R_{\bar{z}\bar{z}}(\tau) = e^{j\omega_0\tau}E[e^{j\lambda\tau \boldsymbol{c}(0)}] \quad \text{approximates} \quad R_{zz}(\tau) = e^{j\omega_0\tau}E[e^{j\lambda\boldsymbol{\varphi}(\tau)}].$$

For zero-mean Gaussian SSS $\boldsymbol{c}(t)$ with

$$R_{cc}(\tau) \approx \begin{cases} \rho, & \text{for } |\tau| < \tau_0; \\ 0, & \text{otherwise.} \end{cases}$$

$\boldsymbol{\varphi}(t) = \int_0^t \boldsymbol{c}(\alpha)d\alpha$ is also zero-mean Gaussian. This implies

$$\begin{aligned} E[\boldsymbol{\varphi}^2(\tau)] &= \int_0^\tau \int_0^\tau E[\boldsymbol{c}(\alpha)\boldsymbol{c}(\beta)]d\alpha d\beta = \int_0^\tau \int_0^\tau R_{cc}(\alpha-\beta)d\alpha d\beta = \int_0^\tau \int_{-\beta}^{\tau-\beta} R_{cc}(u)dud\beta \\ &= \int_{-\tau}^0 R_{cc}(u)\int_{-u}^{\tau} d\beta du + \int_0^\tau R_{cc}(u)\int_0^{\tau-u} d\beta du \quad (\text{Note } R_{cc}(u) = R_{cc}(-u)) \\ &= 2\int_0^\tau (\tau-u)R_{cc}(u)du = \begin{cases} \rho\tau^2, & \text{if } |\tau| < \tau_0; \\ \rho\tau_0(2|\tau|-\tau_0), & \text{otherwise,} \end{cases} \end{aligned}$$

and for zero-mean Gaussian $\boldsymbol{\varphi}(t)$,

$$R_{zz}(\tau) = e^{j\omega_0\tau}e^{-\frac{1}{2}\lambda^2 E[\boldsymbol{\varphi}^2(\tau)]} = \begin{cases} e^{j\omega_0\tau}e^{-\frac{1}{2}\lambda^2\rho\tau^2}, & \text{if } |\tau| < \tau_0; \\ e^{j\omega_0\tau}e^{-\frac{1}{2}\lambda^2\rho\tau_0(2|\tau|-\tau_0)}, & \text{otherwise.} \end{cases}$$

Similarly,

$$R_{\bar{z}\bar{z}}(\tau) = e^{j\omega_0\tau}e^{-\frac{1}{2}\lambda^2\tau^2 E[\boldsymbol{c}^2(0)]} = e^{j\omega_0\tau}e^{-\frac{1}{2}\lambda^2\rho\tau^2}.$$

Thus,

$$\begin{aligned}
|S_{zz}(\omega) - S_{\bar{z}\bar{z}}(\omega)| &= \left| \int_{|\tau|\geq\tau_0} \left( e^{-\frac{1}{2}\lambda^2\rho\tau^2} - e^{-\frac{1}{2}\lambda^2\rho\tau_0(2|\tau|-\tau_0)} \right) e^{-j(\omega-\omega_0)\tau} d\tau \right| \\
&\leq \int_{|\tau|\geq\tau_0} \left| \left( e^{-\frac{1}{2}\lambda^2\rho\tau^2} - e^{-\frac{1}{2}\lambda^2\rho\tau_0(2|\tau|-\tau_0)} \right) e^{-j(\omega-\omega_0)\tau} \right| d\tau \\
&= 2\int_{\tau_0}^{\infty} \left( e^{-\frac{1}{2}\lambda^2\rho\tau_0(2\tau-\tau_0)} - e^{-\frac{1}{2}\lambda^2\rho\tau^2} \right) d\tau \\
&= \frac{2}{\lambda^2\rho\tau_0} e^{-\lambda^2\rho\tau_0^2/2} - \frac{2}{\lambda}\sqrt{\frac{2\pi}{\rho}}\Phi\left(-\lambda\sqrt{\rho}\tau_0\right),
\end{aligned}$$

where $\Phi(\cdot)$ is the cdf of the standard normal. A well-known approximation for $\Phi(-x)$ is

$$\Phi(-x) = \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \left( 1 - \frac{1}{x^2} + \frac{1\cdot 3}{x^4} - \frac{1\cdot 3\cdot 5}{x^6} + \cdots \right)$$

and

$$\frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \left( 1 - \frac{1}{x^2} \right) \leq \Phi(-x) \Rightarrow \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} - \Phi(x) \leq \frac{1}{\sqrt{2\pi}x^3} e^{-x^2/2}.$$

Consequently, letting $x = \lambda\sqrt{\rho}\tau_0$ yields

$$|S_{zz}(\omega) - S_{\bar{z}\bar{z}}(\omega)| \leq \frac{2\tau_0\sqrt{2\pi}}{x} \left( \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} - \Phi\left(-x\right) \right) \leq \frac{2\tau_0\sqrt{2\pi}}{x} \frac{1}{\sqrt{2\pi}x^3} e^{-x^2/2} = \frac{2}{\lambda^4\rho^2\tau_0^3} e^{-\lambda^2\rho\tau_0^2/2}.$$

# Examination of Woodward's Approximate



We finally conclude that:

$$
\begin{aligned}
|S_{xx}(\omega) - S_{\bar{x}\bar{x}}(\omega)| &= \left|\frac{1}{4}\left(S_{zz}(\omega) + S_{zz}(-\omega)\right) - \frac{1}{4}\left(S_{\bar{z}\bar{z}}(\omega) + S_{\bar{z}\bar{z}}(-\omega)\right)\right| \\
&\le \frac{1}{4}\left|S_{zz}(\omega) - S_{\bar{z}\bar{z}}(\omega)\right| + \frac{1}{4}\left|S_{zz}(-\omega) - S_{\bar{z}\bar{z}}(-\omega)\right| \\
&\le \frac{\tau_0}{(\lambda^2\rho\tau_0^2)^2} e^{-\lambda^2\rho\tau_0^2/2},
\end{aligned}
$$

and the difference of $S_{xx}(\omega)$ and $S_{\bar{x}\bar{x}}(\omega)$ uniformly decreases to zero as $\lambda$ large.

# Examination of Woodward's Approximate



- *Wideband FM*: If $\lambda$ is chosen such that $\lambda^2\rho\tau_0^2 \gg 1$, $S_{xx}(\omega) \approx S_{\bar{x}\bar{x}}(\omega)$.
  In such case,
  $$S_{zz}(\omega+\omega_0) \approx S_{\bar{z}\bar{z}}(\omega+\omega_0) = \frac{2\pi}{\lambda} f_{\boldsymbol{c}}\left(\frac{\omega}{\lambda}\right), \quad \text{(See Slide 10-23.)}$$
  and the bandwidth of $S_{\bar{z}\bar{z}}(\omega+\omega_0)$ is *wide* (as proportional to $\lambda$), and so is the bandwidth of its approximate target $S_{zz}(\omega+\omega_0)$. Thus, the system is named *wideband FM*.

- *Narrowband FM*: If $\lambda$ is not large enough such that $\lambda^2\rho\tau_0^2 \ll 1$, (and assume $\tau_0$ is very small such that most $|\tau| \geq \tau_0$), then by Slide 10-28,
  $$R_{zz}(\tau) \approx e^{j\omega_0\tau} e^{-\lambda^2\rho\tau_0^2(|\tau|/\tau_0 - 1/2)} \Rightarrow S_{zz}(\omega+\omega_0) \approx \frac{2\lambda^2\rho\tau_0 e^{\lambda^2\rho\tau_0^2/2}}{\omega^2 + \lambda^4\rho^2\tau_0^2}$$
  with 3dB-bandwidth $\omega_{3\text{dB}} = \lambda^2\rho\tau_0$.
  In such case, $S_{zz}(\omega)$ is named *narrowband FM*, and cannot be well-approximated by $S_{\bar{z}\bar{z}}(\omega)$.

**The end of Section 10-3 Modulation**

# 10-4 Cyclostationary Processes



**Cyclostationarity:** A random process $\boldsymbol{x}(t)$ is called *strictly-sense cyclostationary stationary* (SSCS) with period $T$ if its statistical properties are invariant to a shift of the origin by integer multiples of $T$.

**Wide-Sense Cyclostationarity:** A random process $\boldsymbol{x}(t)$ is called *wide-sense cyclostationary stationary* (WSCS) with period $T$ if $\eta_{xx}(t+mT) = \eta_{xx}(t)$ and $R_{xx}(t_1+mT, t_2+mT) = R_{xx}(t_1, t_2)$ for every integer $m$.

**Theorem 10-5 (SSCS and SSS)** If $\boldsymbol{x}(t)$ is an SSCS process with period $T$, then $\boldsymbol{y}(t) = \boldsymbol{x}(t-\boldsymbol{\theta})$ is SSS, where random variable $\boldsymbol{\theta}$ that is independent of $\boldsymbol{x}(t)$ is uniformly distributed over $[0, T)$.

Moreover, the cdf of $\boldsymbol{y}(t)$ can be obtained from the cdf of $\boldsymbol{x}(t)$ as:

$$F_y(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{1}{T}\int_0^T F_x(x_1, \ldots, x_n; t_1 - \alpha, \ldots, t_n - \alpha)d\alpha. \quad (10.4)$$

**Proof:** It suffices to show that the probability of the event

$$P(\{\zeta \in S : \boldsymbol{y}(t_1 + c, \zeta) \le x_1 \text{ and } \cdots \text{ and } \boldsymbol{y}(t_n + c, \zeta) \le x_n\})$$

is independent of $c$, and is given by (10.4). This can be proved as follows.

By the uniformity of $\boldsymbol{\theta}$, and independence between $\boldsymbol{x}(t)$ and $\boldsymbol{\theta}$,

$$\begin{aligned}
&P(\{\zeta \in S : \boldsymbol{y}(t_1 + c, \zeta) \le x_1 \text{ and } \cdots \text{ and } \boldsymbol{y}(t_n + c, \zeta) \le x_n\}) \\
&= \int_0^T P(\{\zeta \in S : \boldsymbol{x}(t_1 + c - \theta, \zeta) \le x_1 \text{ and } \cdots \text{ and } \boldsymbol{x}(t_n + c - \theta, \zeta) \le x_n\}) \left(\frac{1}{T}\right) d\theta \\
&= \frac{1}{T} \int_{-c}^{T-c} P(\{\zeta \in S : \boldsymbol{x}(t_1 - \alpha, \zeta) \le x_1 \text{ and } \cdots \text{ and } \boldsymbol{x}(t_n - \alpha, \zeta) \le x_n\}) d\alpha \quad (\alpha = \theta - c) \\
&= \frac{1}{T} \int_0^T P(\{\zeta \in S : \boldsymbol{x}(t_1 - \alpha, \zeta) \le x_1 \text{ and } \cdots \text{ and } \boldsymbol{x}(t_n - \alpha, \zeta) \le x_n\}) d\alpha \quad (\text{By SSCS of } \boldsymbol{x}(t)) \\
&= \frac{1}{T} \int_0^T F_x(x_1, \ldots, x_n; t_1 - \alpha, \ldots, t_n - \alpha) d\alpha.
\end{aligned}$$

□

**Theorem 10-6 (WSCS and WSS)** If $\boldsymbol{x}(t)$ is a WSCS process with period $T$, then $\boldsymbol{y}(t) = \boldsymbol{x}(t - \boldsymbol{\theta})$ is WSS, where random variable $\boldsymbol{\theta}$ that is independent of $\boldsymbol{x}(t)$ is uniformly distributed over $[0, T)$.

Moreover, the mean and autocorrelation function of $\boldsymbol{y}(t)$ are

$$\eta_y = \frac{1}{T}\int_0^T \eta_x(t)dt \quad \text{and} \quad R_{yy}(\tau) = \frac{1}{T}\int_0^T R_{xx}(t+\tau, t)dt.$$

**Proof:**

$$\begin{aligned}
E[\boldsymbol{y}(t)] &= E[\boldsymbol{x}(t-\boldsymbol{\theta})] = E[E[\boldsymbol{x}(t-\theta)|\boldsymbol{\theta}=\theta]] = \frac{1}{T}\int_0^T E[\boldsymbol{x}(t-\theta)|\boldsymbol{\theta}=\theta]d\theta \\
&= \frac{1}{T}\int_0^T E[\boldsymbol{x}(t-\theta)]d\theta \quad (\text{Independence between } \boldsymbol{x}(t) \text{ and } \boldsymbol{\theta}) \\
&= \frac{1}{T}\int_0^T \eta_x(t-\theta)d\theta \\
&= \frac{1}{T}\int_{t-T}^t \eta_x(s)ds \quad (s = t-\theta) \\
&= \frac{1}{T}\int_0^T \eta_x(s)ds \quad (\text{WSCS of } \boldsymbol{x}(t)),
\end{aligned}$$

and similarly

$$
\begin{aligned}
R_{yy}(t+\tau,t) &= E[\boldsymbol{y}(t+\tau)\boldsymbol{y}(t)] = E[\boldsymbol{x}(t+\tau-\boldsymbol{\theta})\boldsymbol{x}(t-\boldsymbol{\theta})] \\
&= E[E[\boldsymbol{x}(t+\tau-\theta)\boldsymbol{x}(t-\theta)|\boldsymbol{\theta}=\theta]] \\
&= \frac{1}{T}\int_0^T E[\boldsymbol{x}(t+\tau-\theta)\boldsymbol{x}(t-\theta)|\boldsymbol{\theta}=\theta]d\theta \quad \text{(Uniformity of } \boldsymbol{\theta}) \\
&= \frac{1}{T}\int_0^T E[\boldsymbol{x}(t+\tau-\theta)\boldsymbol{x}(t-\theta)]d\theta \quad \text{(Independence between } \boldsymbol{x}(t) \text{ and } \boldsymbol{\theta}) \\
&= \frac{1}{T}\int_0^T R_{xx}(t+\tau-\theta, t-\theta)d\theta \\
&= \frac{1}{T}\int_{t-T}^t R_{xx}(s+\tau, s)ds \quad (s = t-\theta) \\
&= \frac{1}{T}\int_0^T R_{xx}(s+\tau, s)ds \quad \text{(WSCS of } \boldsymbol{x}(t)).
\end{aligned}
$$

□

## Remarks

- In the literature, $R_{yy}(\tau)$ is called the *time-average autocorrelation function* of the WSCS process $\boldsymbol{x}(t)$, because it averages over one period of the periodic autocorrelation function of $\boldsymbol{x}(t)$, and is usually denoted by $\bar{R}_{xx}(\tau)$.

For a non-WSCS process $\boldsymbol{x}(t)$, its *time-average autocorrelation function* is defined as:

$$\bar{R}_{xx}(\tau) \triangleq \lim_{w\to\infty} \frac{1}{2w}\int_{-w}^{w} R_{xx}(t+\tau, t)dt,$$

provided the limit exists.
The above limit always exists for a WSCS process, and is equal to

$$\bar{R}_{xx}(\tau) = \frac{1}{2kT}\int_{-kT}^{kT} R_{xx}(t+\tau, t)dt \quad \text{for any positive integer } k.$$

- In the textbook, $\bar{\boldsymbol{x}}(t) = \boldsymbol{x}(t-\boldsymbol{\theta})$ is named the *shifted process* of $\boldsymbol{x}(t)$.

# Examples of WSCS Processes



**Examples of WSCS processes**: Show that the pulse train

$$\boldsymbol{z}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \delta(t - nT)$$

is WSCS, where $\{\boldsymbol{c}_n\}_{n=-\infty}^{\infty}$ is a discrete-time SSS sequence. Then, determine the *time-average autocorrelation function* and *time-average power spectrum* of $\boldsymbol{z}(t)$.

**Answer:** Apparently,

$$\mu_z(t) = E[\boldsymbol{z}(t)] = \sum_{n=-\infty}^{\infty} E[\boldsymbol{c}_n]\delta(t - nT) = \mu_c \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

and

$$\begin{aligned} R_{zz}(t_1, t_2) &= \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} E[\boldsymbol{c}_n \boldsymbol{c}_r]\delta(t_1 - nT)\delta(t_2 - rT) \\ &= \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} R_{cc}[n - r]\delta(t_1 - nT)\delta(t_2 - rT) \\ &= \sum_{m=-\infty}^{\infty} R_{cc}[m] \sum_{r=-\infty}^{\infty} \delta(t_1 - (m + r)T)\delta(t_2 - rT) \quad (m = n - r) \end{aligned}$$

are periodic with period $T$.

# Examples of WSCS Processes



To determine the time-average autocorrelation function for $\boldsymbol{x}(t)$, we derive that

$$
\begin{aligned}
\bar{R}_{zz}(\tau) &= \frac{1}{T}\int_0^T R_{zz}(t+\tau,t)dt = \frac{1}{T}\int_0^T \sum_{m=-\infty}^{\infty} R_{cc}[m] \sum_{r=-\infty}^{\infty} \delta(t+\tau-(m+r)T)\delta(t-rT)dt \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_{cc}[m] \sum_{r=-\infty}^{\infty} \int_0^T \delta(t+\tau-(m+r)T)\delta(t-rT)dt \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_{cc}[m] \sum_{r=-\infty}^{\infty} \int_{-rT}^{T-rT} \delta(s+\tau-mT)\delta(s)ds \quad (s=t-rT) \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_{cc}[m] \int_{-\infty}^{\infty} \delta(s+\tau-mT)\delta(s)ds \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_{cc}[m] \int_{-\infty}^{\infty} g_{\tau,m}(s)\delta(s)ds \quad (g_{\tau,m}(s)=\delta(s+\tau-mT)) \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_{cc}[m] g_{\tau,m}(0) \quad \text{Replication Property (Slide 9-87): } \begin{cases} g_{\tau,m}(s) \text{ continuous at } s=0; \\ \text{exception occurs at } \ldots \end{cases} \\
&= \frac{1}{T}\sum_{m=-\infty}^{\infty} R_{cc}[m]\delta(\tau-mT).
\end{aligned}
$$

# Examples of WSCS Processes



For any $\tau$ not equal to a multiple of $T$, $g_{\tau,m}(s)$ is zero at the vicinity of zero, and is certainly continuous at $s = 0$.
The same claim holds if $\tau = kT$ for some integer $k$, and $m \neq k$.
At the situation where $\tau = kT$ and $k = m$, we use the "convention" that $\int_{-\infty}^{\infty} \delta(s + mT - mT)\delta(s)ds = \int_{-\infty}^{\infty} \delta(s)\delta(s)ds = \delta(s)$ to complete the derivation.

The expression on page 475 of the textbook, namely,

$$\int_0^T \delta(t + \tau - (m + r)T)\delta(t - rT)dt = \delta(\tau - mT)$$

is incorrect. Note that the left-hand-side is a function of $r$ while the right-hand-side does not depend on $r$. The correct expression should be:

$$\sum_{r=-\infty}^{\infty} \int_0^T \delta(t + \tau - (m + r)T)\delta(t - rT)dt = \delta(\tau - mT).$$

An easier way to do the derivation in the previous slide is that

$$\int_{-\infty}^{\infty} \delta(t - a)\delta(t - b)dt = \int_{-\infty}^{\infty} \delta(a - b)\delta(t - b)dt = \delta(a - b).$$

# Examples of WSCS Processes



The time-average power spectrum of $\boldsymbol{x}(t)$ is then given by:

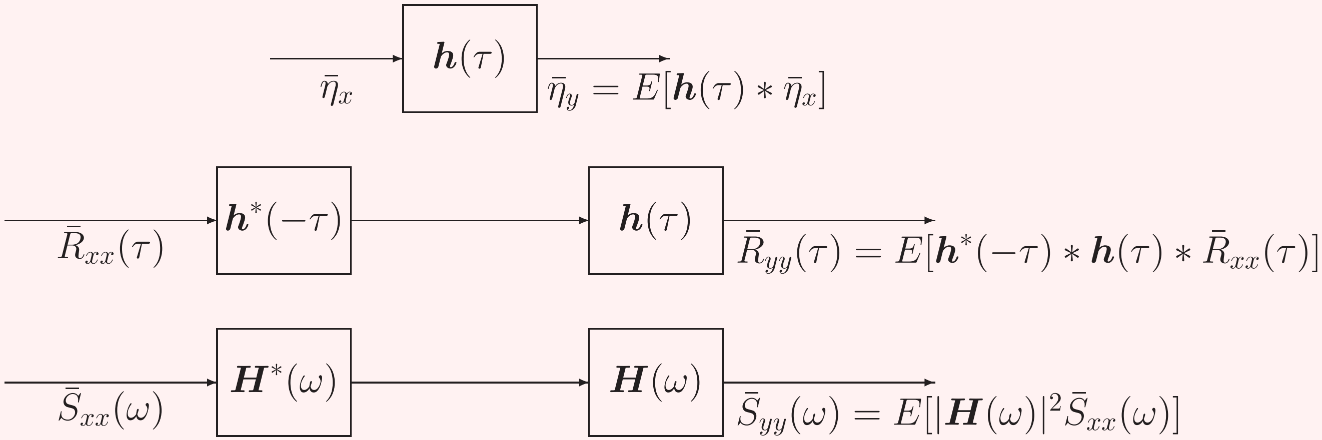
$$\begin{aligned}
\bar{S}_{zz}(\omega) &= \int_{-\infty}^{\infty} \bar{R}_{zz}(\tau) e^{-j\omega\tau} d\tau \\
&= \int_{-\infty}^{\infty} \left( \frac{1}{T} \sum_{m=-\infty}^{\infty} R_{cc}[m] \delta(\tau - mT) \right) e^{-j\omega\tau} d\tau \\
&= \frac{1}{T} \sum_{m=-\infty}^{\infty} R_{cc}[m] \int_{-\infty}^{\infty} \delta(\tau - mT) e^{-j\omega\tau} d\tau \\
&= \frac{1}{T} \sum_{m=-\infty}^{\infty} R_{cc}[m] e^{-j\omega mT} \\
&= \frac{1}{T} S_{cc}[\omega T],
\end{aligned}$$

where

$$S_{cc}[\omega] = \sum_{m=-\infty}^{\infty} R_{cc}[m] e^{-j\omega m}.$$

□

**Fundamental Theorem and Theorems 9-2 and 9-4 Revisited** For any linear time-invariant system,




provided the listed conditions hold.

1. $E[\boldsymbol{P}_h^2] < \infty$, where $\boldsymbol{P}_h \triangleq \int_{-\infty}^{\infty} |\boldsymbol{h}(\tau)| d\tau$;
2. $\limsup_{w\to\infty} \max\left\{\left|\frac{1}{2w}\int_{-w}^{w} \eta_x(t-a)dt\right|, \left|\frac{1}{2w}\int_{-w}^{w} R_{xx}(t-a,t-b)dt\right|\right\} < M$ for some finite $M$ holds almost everywhere (a.e.) in $a, b$;
3. $\lim_{w\to\infty} \frac{1}{2w}\int_{-w}^{w} \eta_x(t-\tau)dt = \bar{\eta}_x$ for every $\tau$;
4. $\lim_{w\to\infty} \frac{1}{2w}\int_{-w}^{w} R_{xx}(t-a,t-b)dt = \bar{R}_{xx}(b-a)$ for every $a, b$.

# Fundamental Thm. and Thms. 9-2 and 9-4 Revisited

**Proof:**

$$\begin{aligned}
\bar{\eta}_y &= \lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}\eta_y(t)dt = \lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}E[\boldsymbol{y}(t)]dt \\
&= \lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}E\left[\int_{-\infty}^{\infty}\boldsymbol{h}(\tau)\boldsymbol{x}(t-\tau)d\tau\right]dt \\
&= \lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}\int_{-\infty}^{\infty}E[\boldsymbol{h}(\tau)]E[\boldsymbol{x}(t-\tau)]d\tau dt \\
&= \lim_{w\to\infty}\int_{-\infty}^{\infty}\left(E[\boldsymbol{h}(\tau)]\frac{1}{2w}\int_{-w}^{w}\eta_x(t-\tau)dt\right)d\tau \\
&= \int_{-\infty}^{\infty}\lim_{w\to\infty}\left(E[\boldsymbol{h}(\tau)]\frac{1}{2w}\int_{-w}^{w}\eta_x(t-\tau)dt\right)d\tau
\end{aligned}$$

This step requires the existence of a function $g(\tau) = M\cdot E[|\boldsymbol{h}(\tau)|]$ such that for sufficiently large $\omega$

$$\left|E[\boldsymbol{h}(\tau)]\frac{1}{2w}\int_{-w}^{w}\eta_x(t-\tau)dt\right| \le g(\tau) \text{ a.e. in } \tau \text{ and } \int_{-\infty}^{\infty}g(\tau)d\tau = M\cdot E[\boldsymbol{P}_h] < \infty.$$

$$= \int_{-\infty}^{\infty}\left(E[\boldsymbol{h}(\tau)]\lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}\eta_x(t-\tau)dt\right)d\tau = \bar{\eta}_x\int_{-\infty}^{\infty}E[\boldsymbol{h}(\tau)]d\tau.$$

# Fundamental Thm. and Thms. 9-2 and 9-4 Revisited 

$$\begin{aligned}
\bar{R}_{yy}(\tau) &= \lim_{w\to\infty} \frac{1}{2w} \int_{-w}^{w} R_{yy}(t+\tau, t)dt \\
&= \lim_{w\to\infty} \frac{1}{2w} \int_{-w}^{w} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\boldsymbol{h}^*(u)\boldsymbol{h}(v)] R_{xx}(t+\tau-v, t-u)dvdudt \\
&= \lim_{w\to\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{1}{2w} \int_{-w}^{w} E[\boldsymbol{h}^*(u)\boldsymbol{h}(v)] R_{xx}(t+\tau-v, t-u)dt \right) dvdu \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lim_{w\to\infty} \left( \frac{1}{2w} \int_{-w}^{w} E[\boldsymbol{h}^*(u)\boldsymbol{h}(v)] R_{xx}(t+\tau-v, t-u)dt \right) dvdu
\end{aligned}$$

There exists function $g(u,v) = M \cdot E[|\boldsymbol{h}^*(u)\boldsymbol{h}(v)|]$ such that for sufficiently large $w$,

$$\left| E[\boldsymbol{h}^*(u)\boldsymbol{h}(v)] \frac{1}{2w} \int_{-w}^{w} R_{xx}(t+\tau-v, t-u)dt \right| \le g(u,v)$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(u,v)dvdu = M \cdot E[\boldsymbol{P}_h^2] < \infty.$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\boldsymbol{h}^*(u)\boldsymbol{h}(v)] \lim_{w\to\infty} \left( \frac{1}{2w} \int_{-w}^{w} R_{xx}(t+\tau-v, t-u)dt \right) dvdu \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\boldsymbol{h}^*(u)\boldsymbol{h}(v)] \bar{R}_{xx}(\tau-v+u)dvdu.
\end{aligned}$$

# Fundamental Thm. and Thms. 9-2 and 9-4 Revisited 

Theorem 9-4 follows immediately from Theorem 9-2; hence, we omit it. □

**Example 10-4** Suppose that

$$\boldsymbol{x}(t) = \sum_{n=-\infty}^{\infty} \boldsymbol{c}_n \delta(t - nT) \text{ with } \{\boldsymbol{c}_n\}_{n=-\infty}^{\infty} \text{ zero-mean i.i.d.}$$

and

$$h(\tau) = \begin{cases} 1, & 0 \le t < T; \\ 0, & \text{otherwise} \end{cases}$$

Please find the time-average autocorrelation function and time-average power spectrum of the output process $\boldsymbol{y}(t)$.

**Answer:** Examine the four conditions as follows.

1. $E[\boldsymbol{P}_h^2] = T^2 < \infty,$ where $\boldsymbol{P}_h \triangleq \int_{-\infty}^{\infty} |\boldsymbol{h}(\tau)| d\tau = \int_0^T d\tau = T$;
2. Since $\eta_x(t) = 0$ and $R_{cc}[m] = E[\boldsymbol{c}_{n+m}\boldsymbol{c}_n^*] = 0$ if $m \neq 0$

$$\begin{aligned} R_{xx}(t_1, t_2) &= \sum_{m=-\infty}^{\infty} R_{cc}[m] \sum_{r=-\infty}^{\infty} \delta(t_1 - (m+r)T)\delta(t_2 - rT) \\ &= R_{cc}[0] \sum_{r=-\infty}^{\infty} \delta(t_1 - rT)\delta(t_2 - rT) \quad \text{(cf. Slide 10-37)} \end{aligned}$$

## Fundamental Thm. and Thms. 9-2 and 9-4 Revisited



$$\begin{aligned}
&\limsup_{w\to\infty} \max\left\{ \underbrace{\left|\frac{1}{2w}\int_{-w}^{w} \eta_x(t-a)dt\right|}_{=0}, \left|\frac{1}{2w}\int_{-w}^{w} R_{xx}(t-a,t-b)dt\right| \right\} \\
&= \limsup_{w\to\infty} \left|\frac{1}{2w}\int_{-w}^{w} R_{cc}[0]\sum_{r=-\infty}^{\infty}\delta(t-a-rT)\delta(t-b-rT)dt\right| \\
&= R_{cc}[0]\delta(a-b)\cdot\limsup_{w\to\infty}\left|\frac{1}{2w}\int_{-w}^{w}\sum_{r=-\infty}^{\infty}\delta(t-b-rT)dt\right| \\
&= R_{cc}[0]\delta(a-b)\cdot\limsup_{w\to\infty}\left|\frac{1}{2w}\left\lceil\frac{2w}{T}\right\rceil\right| \\
&= \frac{1}{T}R_{cc}[0]\delta(a-b);\ \text{(which is bounded a.e. in } a,b)
\end{aligned}$$

3. $\bar{\eta}_x = \lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}\eta_x(t-\tau)dt = 0$ for every $\tau$;

# Fundamental Thm. and Thms. 9-2 and 9-4 Revisited 

4.

$$\begin{aligned}
\bar{R}_{xx}(b-a) &= \lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w} R_{xx}(t-a,t-b)dt \\
&= \lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w} R_{cc}[0]\sum_{r=-\infty}^{\infty}\delta(t-a-rT)\delta(t-b-rT)dt \\
&= R_{cc}[0]\delta(b-a)\lim_{w\to\infty}\frac{1}{2w}\int_{-w}^{w}\sum_{r=-\infty}^{\infty}\delta(t-b-rT)dt \\
&= \frac{1}{T}R_{cc}[0]\delta(b-a)
\end{aligned}$$

for every $a, b$.

By the validity of the four conditions, Fundamental Theorem and Theorems 9-2 and 9-4 give

$$\eta_y = \eta_x\int_{-\infty}^{\infty} h(\tau)d\tau = \eta_x T = 0,$$

# Fundamental Thm. and Thms. 9-2 and 9-4 Revisited 

$$\begin{aligned}
\bar{R}_{yy}(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^*(u)h(v)\bar{R}_{xx}(\tau - v + u)dvdu \\
&= \int_0^T \int_0^T \frac{1}{T}R_{cc}[0]\delta(\tau - v + u)dvdu \quad (0 < v = u + \tau < T) \\
&= \frac{1}{T}R_{cc}[0]\int_0^T \mathbf{1}\{-\tau < u < T - \tau\}du \\
&= R_{cc}[0]\left(1 - \frac{|\tau|}{T}\right)\cdot \mathbf{1}\{|\tau| < T\}.
\end{aligned}$$

and

$$\begin{aligned}
\bar{S}_{yy}(\omega) &= |H(\omega)|^2\bar{S}_{xx}(\omega) = \left|\int_0^T e^{-j\omega\tau}d\tau\right|^2\left(\frac{1}{T}R_{cc}[0]\right) \\
&= \left(\frac{\sin(\omega T/2)}{\omega/2}\right)^2\left(\frac{1}{T}R_{cc}[0]\right) = R_{cc}[0]\frac{4\sin^2(\omega T/2)}{\omega^2 T}.
\end{aligned}$$

□

**The end of Section 10-4 Cyclostationary Processes**

**Definition (Bandlimited processes)** A process $\boldsymbol{x}(t)$ is called *bandlimited* (BL) if $\bar{S}_{xx}(\omega) = 0$ for $|\omega| > \sigma$, and $\bar{R}_{xx}(0) < \infty$.

Most books do not require $\bar{R}_{xx}(0) < \infty$ for the definition of bandlimited processes. Here, we additionally require $\bar{R}_{xx}(0) < \infty$ for theoretical manipulation convenience. See the next lemma.

**Lemma** A bandlimited process $\boldsymbol{x}(t)$ has Taylor expansion (in the MS sense).

**Proof:** Since $\bar{S}_{xx}(\omega)$ is real and non-negative, and

$$\infty > \bar{R}_{xx}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{S}_{xx}(\omega)e^{j\omega\cdot 0}d\omega = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} \bar{S}_{xx}(\omega)d\omega,$$

we have

$$\int_{-\infty}^{\infty} |j\omega|^{2n}\bar{S}_{xx}(\omega)d\omega \le \sigma^{2n}\int_{-\sigma}^{\sigma} \bar{S}_{xx}(\omega)d\omega < \infty.$$

Hence, the inverse Fourier transform of $|j\omega|^{2n}\bar{S}_{xx}(\omega)$ exists, which implies the $n$th derivative of $\bar{R}_{xx}(\tau)$ exists. Specifically, by Theorem 9-4,

$$\bar{R}_{xx}^{(n)}(\tau) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma} (j\omega)^{2n}\bar{S}_{xx}(\omega)e^{j\omega\tau}d\omega.$$ (Hence, $\boldsymbol{x}^{(n)}(t)$ exists in the MS sense.)

Observe that

$$e^{j\omega v} = \sum_{n=0}^{\infty} \frac{v^n}{n!}(j\omega)^n.$$

So, passing the process $\boldsymbol{x}(t)$ via filter $H(\omega) = e^{j\omega v}$ and filter $H(\omega) = \sum_{n=0}^{\infty} \frac{v^n}{n!}(j\omega)^n$ should result in the same output. Accordingly,

$$\boldsymbol{x}(t+v) = \sum_{n=0}^{\infty} \boldsymbol{x}^{(n)}(t)\frac{v^n}{n!} \quad \text{in the MS sense.}$$

□

**Remarks**

- The above lemma indicates that a BL process is very "smooth" since it has derivatives of any order.
- The next lemma shows further that a BL process is not only "smooth" but also "slow-varying in time."

**Lemma** If $\boldsymbol{x}(t)$ is BL (not necessarily a real process as required in the textbook),

$$\lim_{w\to\infty} \frac{1}{2w}\int_{-w}^{w} E\left[|\boldsymbol{x}(t+\tau) - \boldsymbol{x}(t)|^2\right] dt \le \sigma^2\tau^2\bar{R}_{xx}(0),$$

provided the limit exists.

**Proof:** Let $\boldsymbol{y}(t)$ be the output due to input $\boldsymbol{x}(t)$ and filter $H(\omega) = e^{j\omega\tau} - 1$. Then, $\boldsymbol{y}(t) = \boldsymbol{x}(t+\tau) - \boldsymbol{x}(t)$, and

$$\begin{aligned}\lim_{w\to\infty} \frac{1}{2w} \int_{-w}^{w} E\left[|\boldsymbol{x}(t+\tau) - \boldsymbol{x}(t)|^2\right] dt &= \lim_{w\to\infty} \frac{1}{2w} \int_{-w}^{w} E[|\boldsymbol{y}(t)|^2] dt \\ &= \lim_{w\to\infty} \frac{1}{2w} \int_{-w}^{w} R_{yy}(t,t) dt = \bar{R}_{yy}(0).\end{aligned}$$

Theorem 9-4 states that

$$\begin{aligned}\bar{R}_{yy}(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \bar{S}_{xx}(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\sigma}^{\sigma} |e^{j\omega\tau} - 1|^2 \bar{S}_{xx}(\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\sigma}^{\sigma} 4\sin^2\left(\frac{\omega\tau}{2}\right) \bar{S}_{xx}(\omega) d\omega \\ &\leq \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \omega^2\tau^2 \bar{S}_{xx}(\omega) d\omega \quad (|\sin(\theta)| < |\theta|) \\ &\leq \sigma^2\tau^2 \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \bar{S}_{xx}(\omega) d\omega = \sigma^2\tau^2 \bar{R}_{xx}(0).\end{aligned}$$

□

# Stochastic Sampling Theorem



**Theorem 10-9 (Stochastic sampling theorem)** If $\boldsymbol{x}(t)$ is BL, then

$$\boldsymbol{x}(t+\tau) = \sum_{n=-\infty}^{\infty} \boldsymbol{x}(t+nT)\frac{\sin[\sigma(\tau-nT)]}{\sigma(\tau-nT)} \quad \text{(in the MS sense)},$$

where $T = \pi/\sigma$.

**Proof:** By Fourier series,

$$e^{j\omega\tau} = \sum_{n=-\infty}^{\infty} a_{n,\tau} e^{jnT\omega} \text{ for } |\omega| \leq \sigma,$$

where

$$a_{n,\tau} = \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} e^{j\omega\tau} e^{-jnT\omega} d\omega = \frac{\sin[\sigma(\tau-nT)]}{\sigma(\tau-nT)}.$$

Again, passing the BL-to-$\sigma$ process $\boldsymbol{x}(t)$ via filter $H_1(\omega) = e^{j\omega\tau}$ and filter $H_2(\omega) = \sum_{n=-\infty}^{\infty} a_{n,\tau} e^{jnT\omega}$ with $H_1(\omega) = H_2(\omega)$ for $|\omega| \leq \sigma$ (We actually don't care whether $H_1(\omega) = H_2(\omega)$ for $|\omega| > \sigma$. Why?) should result in the same output. Accordingly,

$$\boldsymbol{x}(t+\tau) = \sum_{n=-\infty}^{\infty} a_{n,\tau}\boldsymbol{x}(t+nT).$$

□

# Completeness with Past Samples



- A deterministic BL signal $x(t)$, defined as $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = 0$ for $|\omega| > \sigma$, can be completely determined only when all samples, including **past** samples and **future** samples, are known.

- However, a stochastic BL signal $\boldsymbol{x}(t)$ can be asymptotically determined only with **past** samples!

**Theorem 10-10** Fix *(i)* a BL process $\boldsymbol{x}(t)$ with bandwidth $\sigma$, *(ii)* a number $T_0 < (1/3)\pi/\sigma$, and *(iii)* a constant $\varepsilon > 0$ arbitrarily small. There exists a (sufficiently large) positive integer $n$ and a set of coefficients $\{a_k\}_{k=1}^{n}$ such that

$$\lim_{w\to\infty} \frac{1}{2w} \int_{-w}^{w} E\left[\left|\boldsymbol{x}(t) - \sum_{k=1}^{n} a_k \boldsymbol{x}(t - kT_0)\right|^2\right] dt < \varepsilon.$$

**Proof:** Let $\boldsymbol{y}(t) = \boldsymbol{x}(t) - \sum_{k=1}^{n} a_k \boldsymbol{x}(t - kT_0)$. Then, $\boldsymbol{y}(t)$ is the output due to input $\boldsymbol{x}(t)$ and filter

$$H(\omega) = 1 - \sum_{k=1}^{n} a_k e^{-jkT_0\omega}.$$

# Completeness with Past Samples



Hence,

$$\bar{R}_{yy}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}|H(\omega)|^2\bar{S}_{xx}(\omega)d\tau = \frac{1}{2\pi}\int_{-\sigma}^{\sigma}|H(\omega)|^2\bar{S}_{xx}(\omega)d\tau.$$

This indicates that if

$$|H(\omega)|^2 < \frac{\epsilon}{\bar{R}_{xx}(0)} \text{ for } |\omega| \le \sigma,$$

then

$$\bar{R}_{yy}(0) \le \frac{\epsilon}{\bar{R}_{xx}(0)}\frac{1}{2\pi}\int_{-\sigma}^{\sigma}\bar{S}_{xx}(\omega)d\tau = \epsilon.$$

The availability of such $H(\omega)$ is proved as follows. Let $a_k = -(-1)^k\binom{n}{k}$. Then,

$$H(\omega) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}e^{-jkT_0\omega} = \left(1 - e^{-j\omega T_0}\right)^n,$$

which gives that for $|\omega| \le \sigma$,

$$|H(\omega)|^2 = \left|1 - e^{-j\omega T_0}\right|^n = |2\sin(\omega T_0/2)|^n \to 0 \text{ as } n \to \infty,$$

because

$$\frac{|\omega T_0|}{2} \le \frac{\sigma T_0}{2} < \frac{\pi}{6}.$$

□

# Completeness with Past Samples



## Remarks

- In Chapter 11, we will see that a desire to make $|H(\omega)|^2 = 0$ for $|\omega| \leq \sigma$ will violate the Paley-Wiener condition, which is the sufficient condition for the existence of $H(e^{j\omega})$ with $|H(e^{j\omega})|^2$ equal to a target $S(e^{j\omega})$.

A power spectrum $S[e^{j\omega}]$ (equiv. $S[z]$) can be factorized to $|H[e^{j\omega}]|^2$ (equiv. $H[z]H[1/z]$) if the Paley-Wiener condition

$$\int_{-\pi}^{\pi} |\log S[e^{j\omega}]| d\omega < \infty.$$

is valid.

- Theorem 10-10 is actually valid for any $T_0 < \pi/\sigma$. In the case of $(1/3)\pi/\sigma \leq T_0 < \pi/\sigma$, a different $H(\omega)$ needs to be chosen. For details, you may refer to the Weierstrass approximation theorem or the Fejer-Riesz factorization theorem.

- Theorem 10-11 further increases the sampling period bound from $\pi/\sigma$ to $N\pi/\sigma$, if the samples of the outputs $\boldsymbol{y}_1(t)$, $\boldsymbol{y}_2(t)$, …, $\boldsymbol{y}_N(t)$ of $N$ linear systems $H_1(\omega)$, $H_2(\omega)$, …, $H_N(\omega)$ due to input $\boldsymbol{x}(t)$ are available.

**Theorem 10-11** Fix a BL process $\boldsymbol{x}(t)$ with bandwidth $\sigma$, and a constant $\tau$. Let $c = 2\sigma/N$ and $T_0 = 2\pi/c$. Then,

$$\boldsymbol{x}(t+\tau) = \sum_{n=-\infty}^{\infty} \left[\boldsymbol{y}_1(t+nT_0)p_1(\tau - nT_0) + \cdots + \boldsymbol{y}_N(t+nT_0)p_N(\tau - nT_0)\right]$$

where $\boldsymbol{y}_k(t)$ is the output of the linear system $H_k(\omega)$ due to input $\boldsymbol{x}(t)$, and

$$p_k(\tau) = \frac{1}{c}\int_{-\sigma}^{-\sigma+c} P_k(\omega,\tau)e^{j\omega\tau}d\omega, \qquad \left(P_k(\omega,\tau) = \sum_{n=-\infty}^{\infty} p_k(\tau - nT_0)e^{-j\omega(\tau - nT_0)}\right) \tag{10.5}$$

and $\{H_k(\omega)\}_{k=1}^N$ and $\{P_k(\omega,\tau)\}_{k=1}^N$ are the solutions of

$$\begin{cases} H_1(\omega)P_1(\omega,\tau) + \cdots + H_N(\omega)P_N(\omega,\tau) & = 1 \\ H_1(\omega+c)P_1(\omega,\tau) + \cdots + H_N(\omega+c)P_N(\omega,\tau) & = e^{jc\tau} \\ \cdots\cdots\cdots & \\ H_1(\omega+(N-1)c)P_1(\omega,\tau) + \cdots + H_N(\omega+(N-1)c)P_N(\omega,\tau) & = e^{j(N-1)c\tau} \end{cases} \tag{10.6}$$

for $\omega \in (-\sigma, -\sigma + c]$.

# Papoulis Sampling Expansion

**Proof:**

- It suffices to show that

$$\underbrace{e^{j\omega\tau}}_{\boldsymbol{x}(t+\tau)} = \sum_{n=-\infty}^{\infty} p_1(\tau - nT_0)\underbrace{H_1(\omega)e^{jn\omega T_0}}_{\boldsymbol{y}_1(t+nT_0)} + \cdots + \sum_{n=-\infty}^{\infty} p_N(\tau - nT_0)\underbrace{H_N(\omega)e^{jn\omega T_0}}_{\boldsymbol{y}_N(t+nT_0)} \quad (10.7)$$

for $|\omega| \leq \sigma$ (namely, for $\omega$ in $(-\sigma, -\sigma + c]$, $(-\sigma + c, -\sigma + 2c]$, ..., $(-\sigma + (N-1)c, -\sigma + Nc = \sigma])$.

- Replacing $\omega$ by $(\tilde{\omega} + kc)$ for the right-hand-side of (10.7) yields:

$$
\begin{aligned}
& H_1(\tilde{\omega}+kc)\sum_{n=-\infty}^{\infty} p_1(\tau - nT_0)e^{jn(\tilde{\omega}+kc)T_0} + \cdots + H_N(\tilde{\omega}+kc)\sum_{n=-\infty}^{\infty} p_N(\tau - nT_0)e^{jn(\tilde{\omega}+kc)T_0} \\
= \; & H_1(\tilde{\omega}+kc)\sum_{n=-\infty}^{\infty} p_1(\tau - nT_0)e^{jn\tilde{\omega}T_0} + \cdots + H_N(\tilde{\omega}+kc)\sum_{n=-\infty}^{\infty} p_N(\tau - nT_0)e^{jn\tilde{\omega}T_0} \quad \text{(since } e^{jnkcT_0} = e^{jnk\cdot 2\pi} = 1.) \\
= \; & H_1(\tilde{\omega}+kc)\left(e^{j\tilde{\omega}\tau}P_1(\tilde{\omega},\tau)\right) + \cdots + H_N(\tilde{\omega}+kc)\left(e^{j\tilde{\omega}\tau}P_N(\tilde{\omega},\tau)\right) \quad \text{(by definition of } \{P_\ell(\omega,\tau)\}_{\ell=1}^N \text{ or (10.5))} \\
= \; & e^{j\tilde{\omega}\tau}\left[H_1(\tilde{\omega}+kc)P_1(\tilde{\omega},\tau) + \cdots + H_N(\tilde{\omega}+kc)P_N(\tilde{\omega},\tau)\right] \\
= \; & e^{j\tilde{\omega}\tau}e^{jkc\tau} \quad \text{(by the } (k+1)\text{th equation in (10.6), which is true for } \tilde{\omega} \in (-\sigma, \sigma + c]) \\
= \; & e^{j(\tilde{\omega}+kc)\tau}
\end{aligned}
$$

Therefore, (10.7) is true for $\omega = \tilde{\omega} + kc \in (-\sigma + kc, \sigma + (k+1)c]$ for $0 \leq k < N$.$\Box$

# Papoulis Sampling Expansion



Claim: $p_k(\tau) = \frac{1}{c}\int_{-\sigma}^{-\sigma+c} P_k(\omega,\tau)e^{j\omega\tau}d\omega \Rightarrow P_k(\omega,\tau) = \sum_{n=-\infty}^{\infty} p_k(\tau - nT_0)e^{-j\omega(\tau-nT_0)}$

*Proof:* From (10.6), it can be induced that $P_k(\omega,\tau)$ are periodic with period $T_0$ because $e^{jkc(\tau-nT_0)} = e^{jkc\tau}$. Thus,

$$P_k(\omega,\tau - nT_0) = P_k(\omega,\tau),$$

which implies

$$\begin{aligned} p_k(\tau - nT_0) &= \frac{1}{c}\int_{-\sigma}^{-\sigma+c} P_k(\omega,\tau - nT_0)e^{j\omega(\tau-nT_0)}d\omega \\ &= \frac{1}{c}\int_{-\sigma}^{-\sigma+c} P_k(\omega,\tau)e^{j\omega(\tau-nT_0)}d\omega \\ &= \frac{1}{c}\int_{-\sigma}^{-\sigma+c} P_k(\omega,\tau)e^{j\omega\tau}e^{-jn\omega T_0}d\omega \end{aligned}$$

Accordingly,

$$P_k(\omega,\tau)e^{j\omega\tau} = \sum_{n=-\infty}^{\infty} p_k(\tau - nT_0)e^{jn\omega T_0}.$$

# Papoulis Sampling Expansion



**Remarks**

- For $N = 1$, we have $c = 2\sigma$, $T_0 = \pi/\sigma$,

$$\boldsymbol{x}(t+\tau) = \sum_{n=-\infty}^{\infty} \boldsymbol{y}_1(t+nT_0)p_1(\tau-nT_0) \text{ and } H_1(\omega)P_1(\omega,\tau) = 1 \text{ for } -\sigma < \omega \leq \sigma.$$

Taking $H_1(\omega) = 1$ (hence, $\boldsymbol{y}_1(t) = \boldsymbol{x}(t)$ and

$$p_1(\tau) = \frac{1}{c}\int_{-\sigma}^{-\sigma+c} P_k(\omega,\tau)e^{j\omega\tau}d\omega = \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} e^{j\omega\tau}d\omega)$$

reduces Theorem 10-11 to Theorem 10-9 (cf. Slide 10-51).

**Theorem 10-9 (Stochastic sampling theorem)** If $\boldsymbol{x}(t)$ is BL, then

$$\boldsymbol{x}(t+\tau) = \sum_{n=-\infty}^{\infty} \boldsymbol{x}(t+nT)\frac{\sin[\sigma(\tau-nT)]}{\sigma(\tau-nT)} \quad \text{(in the MS sense)},$$

where $T = \pi/\sigma$.

- For $N > 1$, sampling is performed at every $T_0 = N(\pi/\sigma)$; however, $N$ samples are taken each time. Thus, no saving in "complexity" is obtained.

# Random Sampling



## Observations and motivation

- The relationship between the (discrete) Fourier transform of equidistance samples $\{x[nT]\}$ of a deterministic parent signal $x(t)$ and the parent signal itself is given by

$$X[\omega] \quad \left( = \sum_{n=-\infty}^{\infty} x(nT)e^{-jnT\omega} \right) \quad = \sum_{n=-\infty}^{\infty} X(\omega + 2n\sigma),$$

where $\sigma = \pi/T$ (cf. Slide 9-132).

- The difference $X(\omega) - X[\omega]$ is called *aliasing error*.

- **Question**: How about the Fourier transform of random samples $\{\boldsymbol{t}_n\}$ of $x(t)$, where $\{\boldsymbol{t}_n\}$ is a Poisson point process (cf. Slide 9-47) with average density $\lambda$?

**Lemma** The normalized (discrete) Fourier transform of random samples $\{\boldsymbol{t}_n\}$ of (continuous) $x(t)$, namely,

$$\frac{1}{\lambda} \sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n)e^{-j\omega \boldsymbol{t}_n},$$

is an unbiased estimate of $X(\omega)$.

# Random Sampling



**Proof:** Let $\boldsymbol{z}(t) = \sum_{n=-\infty}^{\infty} \delta(t - \boldsymbol{t}_n)$. Then,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n) e^{-j\omega \boldsymbol{t}_n} &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \left( \sum_{n=-\infty}^{\infty} \delta(t - \boldsymbol{t}_n) \right) dt \\ &= \int_{-\infty}^{\infty} x(t) \boldsymbol{z}(t) e^{-j\omega t} dt.\end{aligned}$$

Hence,

$$\begin{aligned}E\left[ \sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n) e^{-j\omega \boldsymbol{t}_n} \right] &= \int_{-\infty}^{\infty} x(t) E[\boldsymbol{z}(t)] e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} x(t) \lambda e^{-j\omega t} dt = \lambda X(\omega).\end{aligned}$$

□

$E[\boldsymbol{z}(t)] = \dfrac{\partial E[\boldsymbol{x}(t)]}{\partial t} = \dfrac{\partial(\lambda t)}{\partial t} = \lambda$, where $\boldsymbol{x}(t)$ is the Poison process defined in Example 9-5 (cf. Slide 9-48 and Slide 9-98).

**Lemma** Follow the previous lemma. The estimate variance of the unbiased estimator, namely,

$$E\left[\left|\frac{1}{\lambda}\sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n)e^{-j\omega \boldsymbol{t}_n} - X(\omega)\right|^2\right]$$

approaches zero as $\lambda \to \infty$, provided that the energy of $x(t)$, i.e., $\int_{-\infty}^{\infty} x^2(t)dt$, is finite.

**Proof:**

$$\begin{aligned}
E\left[\left|\sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n)e^{-j\omega \boldsymbol{t}_n}\right|^2\right] &= E\left[\left(\int_{-\infty}^{\infty} x(t)\boldsymbol{z}(t)e^{-j\omega t}dt\right)\left(\int_{-\infty}^{\infty} x(s)\boldsymbol{z}(s)e^{j\omega s}ds\right)\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(s)E[\boldsymbol{z}(t)\boldsymbol{z}(s)]e^{-j\omega(t-s)}dtds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(s)R_{zz}(t,s)e^{-j\omega(t-s)}dtds
\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(s)\frac{\partial^2 R_{xx}(t,s)}{\partial t\partial s}e^{-j\omega(t-s)}dtds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(s)\frac{\partial^2(\lambda\min\{t,s\}+\lambda^2 ts)}{\partial t\partial s}e^{-j\omega(t-s)}dtds \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)x(s)\left[\lambda\delta(t-s)+\lambda^2\right]e^{-j\omega(t-s)}dtds \\
&= \lambda\int_{-\infty}^{\infty} x^2(t)dt + \lambda^2|X(\omega)|^2.
\end{aligned}
$$

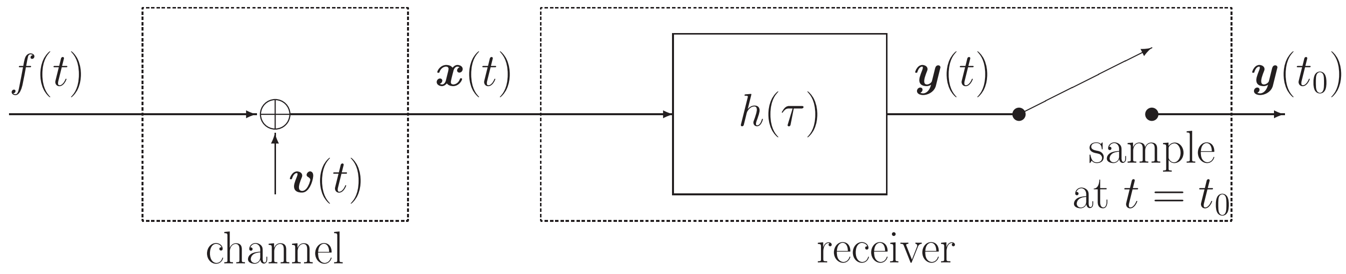
Hence,

$$
\begin{aligned}
E\left[\left|\frac{1}{\lambda}\sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n)e^{-j\omega\boldsymbol{t}_n} - X(\omega)\right|^2\right] &= \frac{1}{\lambda^2}E\left[\left|\sum_{n=-\infty}^{\infty} x(\boldsymbol{t}_n)e^{-j\omega\boldsymbol{t}_n}\right|^2\right] - |X(\omega)|^2 \\
&= \frac{1}{\lambda}\int_{-\infty}^{\infty} x^2(t)dt \to 0 \text{ as } \lambda\to\infty.
\end{aligned}
$$

□

**The end of Section 10-5 Bandlimited Processes and Sampling Theory**

# 10-6 Deterministic Signals in Noise




- A central problem in communications is the *estimation* of a sample $y_f(t_0)$ at a specific time of filter output of a deterministic signal $f(t)$ in presence of noise.
- In absence of noise $\boldsymbol{v}(t)$,

$$\boldsymbol{y}(t_0) = y_f(t_0) \triangleq \int_{-\infty}^{\infty} h(\tau) f(t_0 - \tau) d\tau.$$

- The noise however will change the errorfree system to:

$$\boxed{\boldsymbol{y}(t_0) = y_f(t_0) + \boldsymbol{y}_v(t_0),}$$

where

$$\boldsymbol{y}_v(t_0) \triangleq \int_{-\infty}^{\infty} h(\tau) \boldsymbol{v}(t_0 - \tau) d\tau.$$

**Question:** How to design the filter $h(\tau)$ such that the *output signal-to-noise ratio* $\gamma_o = \dfrac{|y_f(t_0)|^2}{E[\boldsymbol{y}_v^2(t_0)]}$ is maximized, provided the PSD of WSS $\boldsymbol{v}(t)$ is $S_{vv}(\omega)$?

# Matched Filter Principle



**Answer:** The matched filter.

$$
\begin{aligned}
\gamma_o &= \frac{|y_f(t_0)|^2}{E[\boldsymbol{y}_v^2(t_0)]} = \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{vv}(\omega)|H(\omega)|^2 d\omega} \\
&= \frac{\left|\int_{-\infty}^{\infty} F(\omega)S_{vv}^{-1/2}(\omega)\cdot S_{vv}^{1/2}(\omega)H(\omega)e^{j\omega t_0}d\omega\right|^2}{2\pi\int_{-\infty}^{\infty} S_{vv}(\omega)|H(\omega)|^2 d\omega} \\
&\leq \frac{\int_{-\infty}^{\infty}\left|F(\omega)S_{vv}^{-1/2}(\omega)\right|^2 d\omega\cdot\int_{-\infty}^{\infty}\left|S_{vv}^{1/2}(\omega)H(\omega)e^{j\omega t_0}\right|^2 d\omega}{2\pi\int_{-\infty}^{\infty} S_{vv}(\omega)|H(\omega)|^2 d\omega} \quad \text{(Schwartz inequality)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2 S_{vv}^{-1}(\omega)d\omega,
\end{aligned}
$$

with equality holding if, and only if, $k\left(F(\omega)S_{vv}^{-1/2}(\omega)\right)^* = S_{vv}^{1/2}(\omega)H(\omega)e^{j\omega t_0}$ for some complex number $k$, or equivalently $H(\omega) = kF^*(\omega)S_{vv}^{-1}(\omega)e^{-j\omega t_0}$.

# Matched Filter Principle



**Lemma (Schwartz inequality)**

$$\left|\int_a^b f(x)g(x)dx\right|^2 \le \int_a^b |f(x)|^2 dx \int_a^b |g(x)|^2 dx$$

with equality holding if, and only if, $f(x) = kg^*(x)$ for some complex number $k$.

**Proof:** Define $I(a,\theta) \triangleq \int_a^b \left|f(x) - ae^{j\theta}g^*(x)\right|^2 dx$ for real $a \ge 0$ and real $\theta$. Then,

$$\begin{aligned}
I(a,\theta) &= \int_a^b \left|f(x) - ae^{j\theta}g^*(x)\right|^2 dx \\
&= \int_a^b (f(x) - ae^{j\theta}g^*(x))(f^*(x) - ae^{-j\theta}g(x))dx \\
&= \underbrace{\int_a^b |f(x)|^2 dx}_{C} - 2a\,\underbrace{\mathrm{Re}\left\{e^{-j\theta}\int_a^b f(x)g(x)dx\right\}}_{B} + a^2 \underbrace{\int_a^b |g(x)|^2 dx}_{A} \\
&= A\left(a - \frac{B}{A}\right)^2 + \frac{AC - B^2}{A},
\end{aligned}$$

# Matched Filter Principle



where

$$A \triangleq \int_a^b |g(x)|^2 dx,\ B \triangleq \text{Re}\left\{e^{-j\theta}\int_a^b f(x)g(x)dx\right\},\ \text{and } C \triangleq \int_a^b |f(x)|^2 dx.$$

Since $I(a,\theta) \geq 0$ for any $a \geq 0$ and any $\theta$, taking $\theta = \angle \int_a^b f(x)g(x)dx$ such that

$$\begin{aligned} B &= \text{Re}\left\{e^{-j\theta}\int_a^b f(x)g(x)dx\right\} \\ &= \text{Re}\left\{e^{-j\theta}\left|\int_a^b f(x)g(x)dx\right| e^{j\angle \int_a^b f(x)g(x)dx}\right\} = \left|\int_a^b f(x)g(x)dx\right| \geq 0 \end{aligned}$$

yields that $I(B/A,\theta) = (AC - B^2)/A \geq 0$. I.e.,

$$\int_a^b |g(x)|^2 dx \int_a^b |f(x)|^2 dx \geq \left|\int_a^b f(x)g(x)dx\right|^2.$$

It remains to show the sufficiency and necessity of the equality condition.
If $f(x) = kg^*(x)$, equality subsequently holds. On the contrary, if equality holds, then $I(B/A, \angle \int_a^b f(x)g(x)dx) = 0$ implies the desired $k = (B/A)e^{j\angle \int_a^b f(x)g(x)dx}$. □

# Matched Filter Principle



**Special case on matched filter principle**

- When $\boldsymbol{v}(t)$ is white, $S_{vv}(\omega) = N_0/2$.
- $H(\omega) = kF^*(\omega)S_{vv}^{-1}(\omega)e^{-j\omega t_0} = \dfrac{2k}{N_0}F^*(\omega)e^{-j\omega t_0}$ and $k = N_0/2$ implies

$$h(\tau) = f(t_0 - \tau).$$

The system so obtained is called the *matched filter*.

# Tapped Delay Line Approximate of Matched Filter



**Definition (Causal filter)** A causal filter is one whose output depends only on past and present inputs.

Based on this definition, a causal linear time-invariant filter should satisfy $h(\tau) = 0$ for $\tau < 0$.

- $$\begin{aligned} h(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega\tau}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F^*(\omega)}{S_{vv}(\omega)}e^{j\omega(\tau-t_0)}d\omega \\ &= f(-s) * q(s)\big|_{s=\tau-t_0}, \end{aligned}$$

  where $q(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{vv}^{-1}(\omega)e^{j\omega s}d\omega$.

- To convolve $f(-s)$ (even satisfying $f(-s) \neq 0$ only for $-t_0 \leq s \leq 0$) with $q(s)$ may "enlarge" the "non-zero range" of $f(s)$, and hence, may make $h(\tau)$ *unrealistically noncausal*.

- In addition, the resultant $h(\tau)$ may not be *practically realizable*.

- This motivates the development of a *suboptimal* but *directly realizable* alternative filter.

# Tapped Delay Line Approximate of Matched Filter

**Best filter under tapped delay line structure**

- Given that $H(\omega)$ is of the shape:
$$H(\omega) = a_0 + a_1 e^{-j\omega T} + \cdots + a_m e^{-jm\omega T},$$
find the best (real) $\{a_0, a_1, \ldots, a_m\}$ such that $\gamma_o$ is maximized.

**Solution:**

- $y_f(t_0) = \sum_{i=0}^{m} a_i f(t_0 - iT)$ and $\boldsymbol{y}_v(t_0) = \sum_{i=0}^{m} a_i \boldsymbol{v}(t - iT)$.
- To maximize
$$\gamma_o = |y_f(t_0)|^2 / E[\boldsymbol{y}_v^2(t_0)] = c^2 / E[\boldsymbol{y}_v^2(t_0)]$$
is equivalent to the minimization of $E[\boldsymbol{y}_v^2(t_0)]$ subject to $y_f(t_0) = c$ (followed by the maximization with respect to $c$).
- Using the Lagrange multiplier technique, we minimize
$$\begin{aligned} V &\triangleq E[\boldsymbol{y}_v^2(t_0)] - 2\lambda(y_f(t_0) - c) \\ &= \sum_{n=0}^{m}\sum_{i=0}^{m} a_n a_i R_{vv}(nT - iT) - 2\lambda\left(\sum_{n=0}^{m} a_n f(t_0 - nT) - c\right). \end{aligned}$$
Derive
$$\frac{\partial V}{\partial a_n} = \sum_{i=0}^{m} a_i R_{vv}(nT - iT) + \sum_{i=0}^{m} a_i R_{vv}(iT - nT) - 2\lambda f(t_0 - nT) = 0.$$

# Tapped Delay Line Approximate of Matched Filter 

Under $R_{vv}(\tau) = R_{vv}(-\tau)$, this leads to:

$$\mathbb{R}\vec{a} = \lambda \vec{f},$$

where

$$\mathbb{R} = \begin{bmatrix} R_{vv}(0) & R_{vv}(-T) & R_{vv}(-2T) & \cdots & R_{vv}(-mT) \\ R_{vv}(T) & R_{vv}(0) & R_{vv}(-T) & \cdots & R_{vv}(-(m-1)T) \\ R_{vv}(2T) & R_{vv}(T) & R_{vv}(0) & \cdots & R_{vv}(-(m-2)T) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{vv}(mT) & R_{vv}((m-1)T) & R_{vv}((m-2)T) & \cdots & R_{vv}(0) \end{bmatrix},$$

$$\vec{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \quad \text{and} \quad \vec{f} = \begin{bmatrix} f(t_0) \\ f(t_0 - T) \\ f(t_0 - 2T) \\ \vdots \\ f(t_0 - mT) \end{bmatrix}.$$

As a result, $\vec{a}_{\text{opt}} = \lambda \mathbb{R}^{-1} \vec{f}$, where $\lambda$ is chosen such that $\vec{a}_{\text{opt}}^T \vec{f} = c$, namely,

$$\vec{a}_{\text{opt}}^T \vec{f} = \left(\lambda \mathbb{R}^{-1} \vec{f}\right)^T \vec{f} = c \implies \lambda = \frac{c}{\vec{f}^T (\mathbb{R}^{-1})^T \vec{f}}.$$

# Tapped Delay Line Approximate of Matched Filter

With the availability of the result that

$$\vec{a}_{\text{opt}} = c\frac{\mathbb{R}^{-1}\vec{f}}{\vec{f}^T(\mathbb{R}^{-1})^T\vec{f}}$$

we finally obtain:

$$\gamma_o = \frac{y_f^2(t_0)}{E[\boldsymbol{y}_v^2(t_0)]} = \frac{c^2}{\vec{a}_{\text{opt}}^T\mathbb{R}\vec{a}_{\text{opt}}} = \frac{c^2}{c^2\frac{\vec{f}^T(\mathbb{R}^{-1})^T\mathbb{R}\mathbb{R}^{-1}\vec{f}}{\left(\vec{f}^T(\mathbb{R}^{-1})^T\vec{f}\right)^2}} = \vec{f}^T(\mathbb{R}^{-1})^T\vec{f},$$

which is nothing to do with the choice of constant $c$. □

Problem 10-26(b) in the textbook indicates that

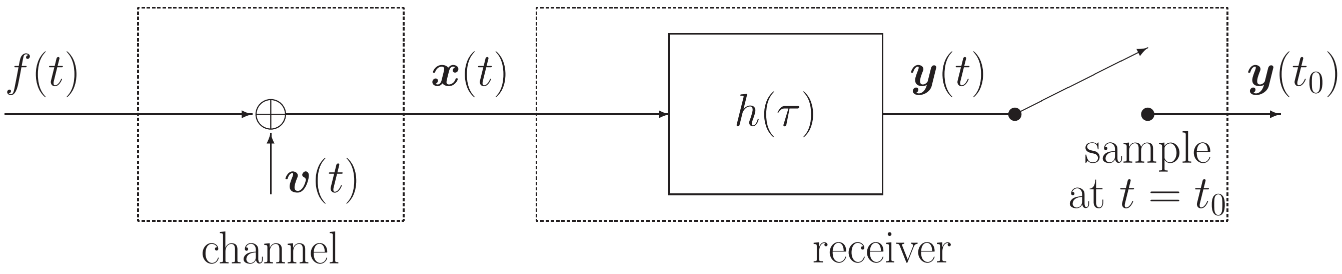
$$\sqrt{\gamma_o} = \sqrt{\frac{y_f(t_0)}{\lambda}}.$$

Since $y_f(t_0) = c$ and $\lambda = c/\vec{f}^T(\mathbb{R}^{-1})^T\vec{f}$,

$$\sqrt{\gamma_o} = \sqrt{\frac{y_f(t_0)}{\lambda}} = \sqrt{\frac{c}{\lambda}} = \sqrt{\vec{f}^T(\mathbb{R}^{-1})^T\vec{f}}.$$

# Smoothing in the MS sense






- A central problem in communications is the *estimation* of a sample $f(t_0)$ at a specific time of filter **input** of a deterministic signal $f(t)$ in presence of noise.
- In absence of noise $\boldsymbol{v}(t)$,

$$\boldsymbol{y}(t_0) = y_f(t_0) \triangleq \int_{-\infty}^{\infty} h(\tau) f(t_0 - \tau) d\tau.$$

- The noise however will change the errorfree system to:

$$\boxed{\boldsymbol{y}(t_0) = y_f(t_0) + \boldsymbol{y}_v(t_0),}$$

where

$$\boldsymbol{y}_v(t_0) \triangleq \int_{-\infty}^{\infty} h(\tau) \boldsymbol{v}(t_0 - \tau) d\tau.$$

**Question:** How to design the filter $h(\tau)$ such that $e \triangleq E\{[\boldsymbol{y}(t_0) - f(t_0)]^2\}$ is minimized, provided $\boldsymbol{v}(t)$ is (possibly time-varying) zero-mean white (i.e., $R_{vv}(t + \tau, t) = q(t)\delta(\tau)$)?

# Smoothing in the MS sense

**Answer:**

- $$\begin{aligned} e &= E\{[\boldsymbol{y}(t_0) - f(t_0)]^2\} \\ &= E\left\{[y_f(t_0) + \boldsymbol{y}_v(t_0) - f(t_0)]^2\right\} \\ &= (y_f(t_0) - f(t_0))^2 + E[\boldsymbol{y}_v^2(t_0)] \qquad (\boldsymbol{y}_v(t) \text{ zero mean}) \\ &= \left(\int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0)\right)^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)E[\boldsymbol{v}(t_0-u)\boldsymbol{v}(t_0-v)]dudv \\ &= \left(\int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0)\right)^2 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)q(t_0-v)\delta(v-u)dudv \\ &= \left(\int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0)\right)^2 + \int_{-\infty}^{\infty} h^2(v)q(t_0-v)dv \\ &= b^2 + \sigma^2, \end{aligned}$$

  where

  bias $b = \int_{-\infty}^{\infty} h(\tau)f(t_0-\tau)d\tau - f(t_0)$ and variance $\sigma^2 = \int_{-\infty}^{\infty} h^2(v)q(t_0-v)dv.$

# Smoothing in the MS sense



**Assumptions**

- $h(t) = 0$ for $|t| > T$, $h(-t) = h(t)$, and $\displaystyle\int_{-T}^{T} h(t)dt = 1$,
- $q(t_0 - v) = N_0/2$. $\quad \boxed{\Rightarrow \sigma^2 = \frac{N_0}{2}\int_{-T}^{T} h^2(v)dv.}$
- $f(t_0 - \tau) = f(t_0) - \tau f'(t_0) + \frac{\tau^2}{2} f''(t_0)$.

$$
\begin{aligned}
b &= \int_{-\infty}^{\infty} h(\tau)\left[f(t_0) - \tau f'(t_0) + \frac{\tau^2}{2} f''(t_0)\right] d\tau - f(t_0) \\
&= \left(f(t_0)\int_{-T}^{T} h(\tau)d\tau - f(t_0)\right) - f'(t_0)\int_{-T}^{T} \tau h(\tau)d\tau + \frac{f''(t_0)}{2}\int_{-T}^{T} \tau^2 h(\tau)d\tau \\
&= \frac{f''(t_0)}{2}\int_{-T}^{T} \tau^2 h(\tau)d\tau.
\end{aligned}
$$

# Smoothing in the MS sense



By Lagrange multiplier technique, we minimize $e$ subject to

$$\int_{-T}^{T} h(\tau)d\tau = 1 \quad \text{and} \quad \int_{-T}^{T} \tau^2 h(\tau)d\tau = c,$$

and obtain:

$$\begin{aligned}
\frac{\partial e}{\partial h(v)} &= \frac{\partial\left[\frac{[f''(t_0)]^2}{4}c^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau)d\tau - \lambda_1\left(\int_{-T}^{T} h(\tau)d\tau - 1\right) - \lambda_2\left(\int_{-T}^{T} \tau^2 h(\tau)d\tau - c\right)\right]}{\partial h(v)} \\
&= N_0 h(v) - \lambda_1 - \lambda_2 v^2 = 0.
\end{aligned}$$

This implies $h_{\text{opt}}(v) = \frac{1}{N_0}\left(\lambda_1 + \lambda_2 v^2\right)$ for $|v| \le T$.

Some people may be dubious about (or have troubles to understand) how we can take partial derivative onto $e$ with respect to $h(v)$. Here, I provide an alternative approach to determine the optimal $h_{\text{opt}}(v)$.

$$e = \frac{[f''(t_0)]^2}{4}c^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau)d\tau - \lambda_1\left(\int_{-T}^{T} h(\tau)d\tau - 1\right) - \lambda_2\left(\int_{-T}^{T} \tau^2 h(\tau)d\tau - c\right)$$

# Smoothing in the MS sense



$$= \int_{-T}^{T} \left( \frac{N_0}{2} h^2(\tau) - \left(\lambda_1 + \lambda_2 \tau^2\right) h(\tau) \right) d\tau + \frac{[f''(t_0)]^2}{4} c^2 + \lambda_1 + \lambda_2 c$$

$$= \int_{-T}^{T} \left( \frac{N_0}{2} \left[ h(\tau) - \frac{(\lambda_1 + \lambda_2 \tau^2)}{N_0} \right]^2 - \frac{(\lambda_1 + \lambda_2 \tau)^2}{2N_0} \right) d\tau + \frac{[f''(t_0)]^2}{4} c^2 + \lambda_1 + \lambda_2 c$$

Apparently, choosing $h(\tau)$ other than $\frac{(\lambda_1+\lambda_2\tau^2)}{N_0}$ can only grow $e$. Thus, $h_{\text{opt}}(\tau) = \frac{(\lambda_1+\lambda_2\tau^2)}{N_0}$.

Solving

$$\int_{-T}^{T} h_{\text{opt}}(\tau) d\tau = \frac{2T}{N_0}\lambda_1 + \frac{2T^3}{3N_0}\lambda_2 = 1 \quad \text{and} \quad \int_{-T}^{T} \tau^2 h_{\text{opt}}(\tau) d\tau = \frac{2T^3}{3N_0}\lambda_1 + \frac{2T^5}{5N_0}\lambda_2 = c$$

yields

$$\lambda_1 = -\frac{15N_0}{8T^3}\left(c - \frac{3}{5}T^2\right) \quad \text{and} \quad \lambda_2 = \frac{45N_0}{8T^5}\left(c - \frac{1}{3}T^2\right),$$

and

$$h_{\text{opt}}(v) = \frac{15}{8T}\left[\left(3\frac{c}{T^2} - 1\right)\frac{v^2}{T^2} - \left(\frac{c}{T^2} - \frac{3}{5}\right)\right].$$

# Smoothing in the MS sense



The textbook also requires that $h(t) > 0$ for $|t| \leq T$.
By examining $h_{\text{opt}}(0) = \frac{15}{8T}\left(\frac{3}{5} - \frac{c}{T^2}\right) > 0$ and $h_{\text{opt}}(T) = \frac{15}{4T}\left(\frac{c}{T^2} - \frac{1}{5}\right) > 0$, this requirement is equivalent to

$$\frac{3}{5} > \frac{c}{T^2} > \frac{1}{5}.$$

By letting $\bar{c} = c/T^2$ and $\bar{\tau} = \tau/T$, we derive:

$$\begin{aligned}
e &= \frac{[f''(t_0)]^2}{4}c^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau)d\tau \\
&= \frac{[f''(t_0)]^2 T^4}{4}\bar{c}^2 + \frac{N_0 T}{2}\int_{-1}^{1} h^2(T\bar{\tau})d\bar{\tau} \\
&= \frac{[f''(t_0)]^2 T^4}{4}\bar{c}^2 + \frac{225 N_0}{128 T}\int_{-1}^{1} [(3\bar{c} - 1)\bar{\tau}^2 - (\bar{c} - 3/5)]^2 d\bar{\tau} \\
&= \frac{[f''(t_0)]^2 T^4}{4}\bar{c}^2 + \frac{3N_0}{16T}(3 - 10\bar{c} + 15\bar{c}^2) \\
&= \left(\frac{[f''(t_0)]^2 T^4}{4} + \frac{45 N_0}{16T}\right)\bar{c}^2 - \frac{15 N_0}{8T}\bar{c} + \frac{9N_0}{16T}.
\end{aligned}$$

# Smoothing in the MS sense

Consequently,

$$\bar{c}_{\min} = \frac{\dfrac{15N_0}{8T}}{2\left(\dfrac{[f''(t_0)]^2T^4}{4} + \dfrac{45N_0}{16T}\right)} = \frac{15/4}{[f''(t_0)]^2T^5/N_0 + 45/4}$$

and

$$e_{\min} = \frac{9N_0}{16T} \cdot \frac{[f''(t_0)]^2T^5/N_0 + 5}{[f''(t_0)]^2T^5/N_0 + 45/4}.$$

Finally,

$$\begin{aligned} h_{\mathrm{opt}}(v) &= \frac{15}{8T}\left[(3\bar{c}_{\min} - 1)\frac{v^2}{T^2} - \left(\bar{c}_{\min} - \frac{3}{5}\right)\right] \\ &= \frac{15}{8T}\left[-\frac{A}{(A+45/4)}\frac{v^2}{T^2} + \frac{3(A+5)}{5(A+45/4)}\right] \\ &= \frac{3A}{8T(A+45/4)}\left(-5\frac{v^2}{T^2} + 3 + \frac{15}{A}\right), \end{aligned}$$

where $A = [f''(t_0)]^2T^5/N_0$. □

# Smoothing in the MS sense



This filter does not satisfy $h(t) > 0$ for $|t| \leq T$ since it may happen that $-2 + 15/A < 0$. An advantage of this design is

$$e_{\text{min}} = \frac{9N_0}{16T} \cdot \frac{A+5}{A+45/4} = O(1/T) \to 0 \text{ as } T \to \infty.$$

This is different from the design from textbook (satisfying $h(t) > 0$ for $|t| \leq T$) in which there exists $0 < T_{\text{min}} < \infty$ that minimizes $e$. Such behavior can also be observed in the subsequent *moving average* estimator.

# Moving Average Estimator



Let $h(\tau) = 1/(2T)$ for $|\tau| \le T$, and zero, otherwise.

Then,

$$\begin{aligned} e &= b^2 + \sigma^2 \\ &= \frac{[f''(t_0]^2}{4}\left(\int_{-T}^{T} \tau^2 h(\tau) d\tau\right)^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau) d\tau \\ &= \frac{[f''(t_0]^2}{4}\left(\frac{1}{2T}\int_{-T}^{T} \tau^2 d\tau\right)^2 + \frac{N_0}{2}\int_{-T}^{T} \frac{1}{4T^2} d\tau \\ &= \frac{[f''(t_0]^2 T^4}{36} + \frac{N_0}{4T}, \end{aligned}$$

and

$$\frac{\partial e}{\partial T} = \frac{[f''(t_0)]^2}{9} T^3 - \frac{N_0}{4} T^{-2} = 0$$

implies that

$$T_{\min}^{\text{mae}} = \left(\frac{9N_0}{4[f''(t_0)]^2}\right)^{1/5} \quad \text{and} \quad e_{\min}^{\text{mae}} = \frac{[f''(t_0]^2}{36} \frac{9N_0}{4[f''(t_0]^2 T_{\min}^{\text{mae}}} + \frac{N_0}{4T_{\min}^{\text{mae}}} = \frac{5N_0}{16T_{\min}^{\text{mae}}}.$$

# Parabolic Window Estimator



Let $h(\tau) = \frac{3}{4T}\left(1 - \frac{\tau^2}{T^2}\right)$ for $|\tau| \le T$, and zero, otherwise.

Then,

$$\begin{aligned} e &= b^2 + \sigma^2 \\ &= \frac{[f''(t_0)]^2}{4}\left(\int_{-T}^{T} \tau^2 h(\tau) d\tau\right)^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau) d\tau \\ &= \frac{[f''(t_0)]^2 T^4}{100} + \frac{3N_0}{10T}, \end{aligned}$$

and

$$\frac{\partial e}{\partial T} = \frac{[f''(t_0)]^2}{25} T^3 - \frac{3N_0}{10} T^{-2} = 0$$

implies that

$$T_{\min}^{\text{pwe}} = \left(\frac{15N_0}{2[f''(t_0)]^2}\right)^{1/5} \quad \text{and} \quad e_{\min}^{\text{pwe}} = \frac{[f''(t_0)]^2}{100}\frac{15N_0}{2[f''(t_0)]^2 T_{\min}^{\text{pwe}}} + \frac{3N_0}{10T_{\min}^{\text{pwe}}} = \frac{3N_0}{8T_{\min}^{\text{pwe}}}.$$

# Comparison of Three Estimators



For $|\tau| \le 1$, define

$$w(\tau) \triangleq T \cdot h(T\tau) = \begin{cases} \dfrac{15A}{8(A+45/4)}\left(\dfrac{3}{5}+\dfrac{3}{A}-\tau^2\right), & \text{optimal} \\ \dfrac{15}{8}\left(\dfrac{3}{5}-\tau^2\right), & \textbf{near-optimal} \\ \dfrac{1}{2}, & \text{moving average} \\ \dfrac{3}{4}(1-\tau^2), & \text{parabolic window} \end{cases}$$

and

$$e_{\min} = \begin{cases} \dfrac{9N_0}{16T}\dfrac{A+5}{A+45/4}, & \text{optimal} \\ \dfrac{9N_0}{16T}, & \textbf{near-optimal} \\ \dfrac{5 \cdot 4^{1/5}[f''(t_0)]^{2/5}N_0^{4/5}}{16 \cdot 9^{1/5}} \approx 0.266[f''(t_0)]^{2/5}N_0^{4/5}, & \text{moving average} \\ \dfrac{3 \cdot 2^{1/5}[f''(t_0)]^{2/5}N_0^{4/5}}{8 \cdot 15^{1/5}} \approx 0.251[f''(t_0)]^{2/5}N_0^{4/5}, & \text{parabolic window} \end{cases}$$

where $A = [f''(t_0)]^2T^5/N_0$.

# Near-Optimal Estimator



$$h_{\text{opt}}(v) = \frac{15}{8T}\frac{A}{(A+45/4)}\left(\frac{3}{5} - \frac{v^2}{T^2} + \frac{3}{A}\right),$$

where $A = [f''(t_0)]^2 T^5 / N_0$.

Let $h(\tau) = \frac{15}{8T}\left(\frac{3}{5} - \frac{\tau^2}{T^2}\right)$ for $|\tau| \leq T$, and zero, otherwise.

Then,

$$\begin{aligned} e &= b^2 + \sigma^2 \\ &= \frac{[f''(t_0)]^2}{4}\left(\int_{-T}^{T} \tau^2 h(\tau) d\tau\right)^2 + \frac{N_0}{2}\int_{-T}^{T} h^2(\tau) d\tau \\ &= \frac{[f''(t_0)]^2 T^4}{4}\left(\int_{-1}^{1} \tau^2 w(\tau) d\tau\right)^2 + \frac{N_0}{2T}\int_{-1}^{1} w^2(\tau) d\tau \\ &= 0 + \frac{9N_0}{16T}. \end{aligned}$$

This is an unbiased estimator with asymptotic zero variance!

**The end of Section 10-6 Deterministic Signals in Noise**

# Appendix 10A The Poisson Sum Formula



**Lemma (Poisson sum formula)** For any positive $c$,

$$\sum_{n=-\infty}^{\infty} f(x+nc) = \frac{1}{c}\sum_{n=-\infty}^{\infty} F(nu_0)e^{jnu_0x}$$

where $F(u) = \int_{-\infty}^{\infty} f(x)e^{-jux}dx$ is the Fourier transform of $f(x)$, and $u_0 = 2\pi/c$.

On Slide 9-131∼9-132, we respectively obtain

$$S_{xx}[\omega] = \sum_{n=-\infty}^{\infty} R_{xx}(n)e^{j\omega n} \quad \text{and} \quad S_{xx}[\omega] = \sum_{n=-\infty}^{\infty} S_{xx}(\omega+2n\pi).$$

Hence,

$$\sum_{n=-\infty}^{\infty} S_{xx}(\omega+2n\pi) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} 2\pi R_{xx}(n)e^{jn\omega},$$

where $R_{xx}(\tau) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega$.

# Appendix 10A The Poisson Sum Formula



Let $f(\omega) = S_{xx}(\omega)$ be real and symmetric,
which implies $R_{xx}(\tau)$ is also real and symmetric.
Then,

$$F(u) = \int_{-\infty}^{\infty} f(\omega) e^{-ju\omega} d\omega = \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega(-u)} d\omega = 2\pi R_{xx}(-u) = 2\pi R_{xx}(u),$$

and (with $c = 2\pi$)

$$\sum_{n=-\infty}^{\infty} f\left(x + n(2\pi)\right) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi}{2\pi} n\right) e^{jn(2\pi/(2\pi))x}.$$

Poisson sum formula is an extension of this result by replacing $2\pi$ by $c$, which leads to:

$$\sum_{n=-\infty}^{\infty} f(x + nc) = \frac{1}{c} \sum_{n=-\infty}^{\infty} F\left(\frac{2\pi}{c} n\right) e^{jn(2\pi/c)x}.$$

**Operational meaning:**

**The Inverse Fourier transform of samples causes aliasing**.

# Appendix 10A The Poisson Sum Formula



**Fourier series:** For a periodic function $g$ with period $T_0$,

$$g(x) = \sum_{m=-\infty}^{\infty} c_m e^{jm\omega_0 x}$$

where $\omega_0 = 2\pi/T_0$ and $c_m = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(x)e^{-jm\omega_0 x}dx$.

**Proof of Poisson Sum Formula:** Fourier series said that for $T_0 = c$, $\omega_0 = u_0$, and $g(x) = \sum_{n=-\infty}^{\infty} \delta(x + nc)$,

$$c_m = \frac{1}{c}\int_{-c/2}^{c/2} \sum_{n=-\infty}^{\infty} \delta(x + nc)e^{-jmu_0 x}dx = \frac{1}{c}$$

and

$$g(x) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{c}\right) e^{jnu_0 x}.$$

Hence,

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} f(x+nc) &= f(x) * \left( \sum_{n=-\infty}^{\infty} \delta(x+nc) \right) = f(x) * \left( \frac{1}{c} \sum_{n=-\infty}^{\infty} e^{jnu_0 x} \right) \\
&= \frac{1}{c} \sum_{n=-\infty}^{\infty} \left( f(x) * e^{jnu_0 x} \right) \\
&= \frac{1}{c} \sum_{n=-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(\alpha) e^{jnu_0 (x-\alpha)} d\alpha \right) \\
&= \frac{1}{c} \sum_{n=-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(\alpha) e^{-jnu_0 \alpha} d\alpha \right) e^{jnu_0 x} \\
&= \frac{1}{c} \sum_{n=-\infty}^{\infty} F(nu_0) e^{jnu_0 x}.
\end{aligned}
$$

□