

Chapter 11 Spectral Representation

Po-Ning Chen, Professor

Institute of Communications Engineering

National Chiao Tung University

Hsin Chu, Taiwan 30010, R.O.C.

11-1 Factorization and Innovations

11-1

Concern (for continuous-time processes)

- How to represent a real WSS process $\mathbf{x}(t)$ as a response of a *minimum-phase* system $\mathbf{L}(\omega)$ with a white input $\mathbf{i}(t)$ of unit power?

Definition (Minimum-phase system) A system is called *minimum-phase* if both $\mathbf{L}(\omega)$ and $1/\mathbf{L}(\omega)$ are causal and stable.

(A system is stable if a bounded input (BI) always induces a bounded output (BO). As a result, a linear system is stable in the BIBO sense if all poles of the system are in the strict left half of the s -plane.)

Definition (Causal filter) An causal filter is one whose output depends only on past and present inputs.

- A process that can be represented as a response of a *minimum-phase* system $\mathbf{L}(\omega)$ with a white input $\mathbf{i}(t)$ of unit power is called *regular*.

Oxford Dictionary - **Regular**. adj. Recurring at uniform intervals. Done or happening frequently.

- A formal definition of regular processes is given below.

Definition (Regular processes) A process $\mathbf{x}(t)$ is **regular** if

$$S_{xx}(\omega) = |\mathbf{L}(\omega)|^2$$

where $\mathbf{L}(s)$ ($s = j\omega$) is **analytic** in the right-hand plane $\text{Re}\{s\} > 0$.

- Roughly speaking, a function is **analytic** if its function values are determinate and finite (never indeterminate or infinity).

11-1 Factorization and Innovations

11-2

Filter with minimum group delay

- For all causal and stable systems that have the same magnitude response, the minimum phase system has the *minimum group delay*.
- Hence, a more appropriate name for *minimum-phase system* should be the “*minimum group delay*” system.
- We will come back to (provide a proof for) this later.

Some observations about regular process $\mathbf{x}(t)$

- $R_{ii}(\tau) = \delta(\tau) \Rightarrow S_{ii}(\omega) = 1$.
- $S_{xx}(\omega) = |\mathbf{L}(\omega)|^2 S_{ii}(\omega) = |\mathbf{L}(\omega)|^2$.
- So,

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} \mathbf{1}(\tau) \mathbf{i}(t - \tau) d\tau,$$

where $\mathbf{L}(\omega)$ is *minimum-phase*, which is determined in terms of the desired *real, positive, even, finite-area* $S_{xx}(\omega)$, and $\mathbf{1}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{L}(\omega) e^{j\omega\tau} d\omega$.

Terminologies and Paley-Wiener Condition

11-3

Innovation: $i(t)$ is called the *innovation* of $\mathbf{x}(t)$.

Innovation Filter: $L(\omega)$ is called the *innovation filter* of $\mathbf{x}(t)$.

Whitening Filter: $1/L(\omega)$ is called the *whitening filter* of $\mathbf{x}(t)$.

Lemma (Paley-Wiener condition) A process $\mathbf{x}(t)$ is regular if the Paley-Wiener condition holds, i.e.,

$$\int_{-\infty}^{\infty} \frac{|\log S_{xx}(\omega)|}{1 + \omega^2} d\omega < \infty.$$

- Hence, a BL process violates the Paley-Wiener condition.
- Paley-Wiener condition is only **sufficient**.
- We thus cannot prove [that a BL process is not regular](#) by showing it violates the Paley-Wiener condition.

Definition (Bandlimited processes) A process $\mathbf{x}(t)$ is called *bandlimited* (BL) if $\bar{S}_{xx}(\omega) = 0$ for $|\omega| > \sigma$, and $\bar{R}_{xx}(0) < \infty$.

Rational Spectra

11-4

How to find $L(\omega)$ such that $|L(\omega)|^2 = S(\omega)$ for a given **real, positive, even, finite-area** $S(\omega)$.

- Observation 1: $S(\omega) = S(-\omega)$ implies that $S(\omega)$ is a function of ω^2 .
- Observation 2: $L(\omega)$ can be easily determined if $S(\omega)$ is a rational spectrum.

Definition (Rational spectra) A rational spectrum is the ratio of two polynomials in ω^2 :

$$S(\omega) = \frac{A(\omega^2)}{B(\omega^2)},$$

where $A(x)$ and $B(x)$ are both polynomials of x .

- Let $s = j\omega$. Then, $S(s) = A(-s^2)/B(-s^2)$.
- Observe that if s_i is a root (either zero or pole) of $S(s)$, $-s_i$ is also a root of $S(s)$. Also, the roots of $S(s)$ are either real or complex conjugate.
- Then, roots of $S(s)$ are symmetric with respect to the **imaginary axis**. So we can separate them into two groups: **Left** group that consists of all roots with $\text{Re}\{s_i\} < 0$, and the **right** group that consists of all roots with $\text{Re}\{s_i\} > 0$. (How to take care of those roots with $\text{Re}\{s_i\} = 0$?)
- We can accordingly form $L(s)$ by the ratio of two polynomials with the left roots of $S(s)$.

Rational Spectra

11-5

$$\text{Example 11-1 } S(\omega) = \frac{N}{\alpha^2 + \omega^2}.$$

$$\begin{aligned} \text{Solution: } S(s) &= \frac{N}{\alpha^2 - s^2} = \frac{N}{(|\alpha| + s)(|\alpha| - s)} \Rightarrow L(s) = \frac{\sqrt{N}}{|\alpha| + s} \\ \Rightarrow L(\omega) &= \frac{\sqrt{N}}{|\alpha| + j\omega} \quad \left(\Rightarrow |L(\omega)|^2 = \left| \frac{\sqrt{N}}{|\alpha| + j\omega} \right|^2 = \frac{N}{\alpha^2 + \omega^2} = S(\omega) \right) \end{aligned}$$

□

$$\text{Example 11-2 } S(\omega) = \frac{49 + 25\omega^2}{9 + 10\omega^2 + \omega^4}.$$

$$\begin{aligned} \text{Solution: } S(s) &= \frac{49 - 25s^2}{9 - 10s^2 + s^4} = \frac{(7 + 5s)(7 - 5s)}{(1 + s)(3 + s)(1 - s)(3 - s)} \\ \Rightarrow L(s) &= \frac{7 + 5s}{(1 + s)(3 + s)} \quad \left(S(s) = L(s)L(-s) \right) \\ \Rightarrow L(\omega) &= \frac{7 + 5j\omega}{(1 + j\omega)(3 + j\omega)} \\ \left(\Rightarrow |L(\omega)|^2 &= \left| \frac{7 + 5j\omega}{(1 + j\omega)(3 + j\omega)} \right|^2 = \frac{49 + 25\omega^2}{9 + 10\omega^2 + \omega^4} = S(\omega) \right) \end{aligned}$$

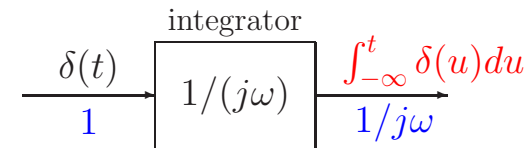
□

Minimum Phase System Revisited

Example 11-1

$$\begin{aligned}
 \mathbf{1}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{L}(\omega) e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{N}}{|\alpha| + j\omega} e^{j\omega\tau} d\omega \\
 &= \sqrt{N} e^{-|\alpha|\tau} \int_{-\infty}^{\tau} \delta(u) du \\
 &= \begin{cases} \sqrt{N} e^{-|\alpha|\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sqrt{N}}{|\alpha| + j\omega} e^{j\omega\tau} d\omega &= \\
 \sqrt{N} e^{-|\alpha|\tau} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{j\omega'} e^{j\omega'\tau} d\omega' \right) & \\
 \text{for } j\omega' = |\alpha| + j\omega. &
 \end{aligned}$$



Example 11-2

$$\begin{aligned}
 \mathbf{1}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{L}(\omega) e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{7 + 5j\omega}{(1 + j\omega)(3 + j\omega)} e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{1}{1 + j\omega} + \frac{4}{3 + j\omega} \right) e^{j\omega\tau} d\omega \\
 &= \begin{cases} e^{-\tau} + 4e^{-3\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases}
 \end{aligned}$$

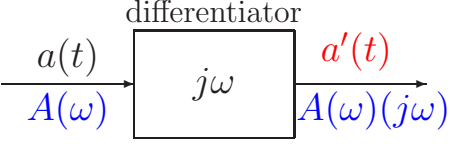
Minimum Phase System Revisited

11-7

Example 11-1

$$\begin{aligned}
 \mathbf{l}_{\text{whitening}}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\mathbf{L}(\omega)} e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|\alpha| + j\omega}{\sqrt{N}} e^{j\omega\tau} d\omega \\
 &= \frac{|\alpha|}{\sqrt{N}} \delta(\tau) + \frac{1}{2\pi\sqrt{N}} \int_{-\infty}^{\infty} (j\omega) e^{j\omega\tau} d\omega
 \end{aligned}$$

differentiator



Example 11-2

$$\begin{aligned}
 \mathbf{l}_{\text{whitening}}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\mathbf{L}(\omega)} e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(1 + j\omega)(3 + j\omega)}{7 + 5j\omega} e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{5} \left(\frac{3 + 4(j\omega) + (j\omega)^2}{1.4 + j\omega} \right) e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{5} \left(-\frac{0.64}{1.4 + j\omega} + 2.6 + j\omega \right) e^{j\omega\tau} d\omega \\
 &= -0.128e^{-1.4\tau} \mathbf{1} \{ \tau > 0 \} + 0.52 \delta(\tau) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{5} (j\omega) e^{j\omega\tau} d\omega
 \end{aligned}$$

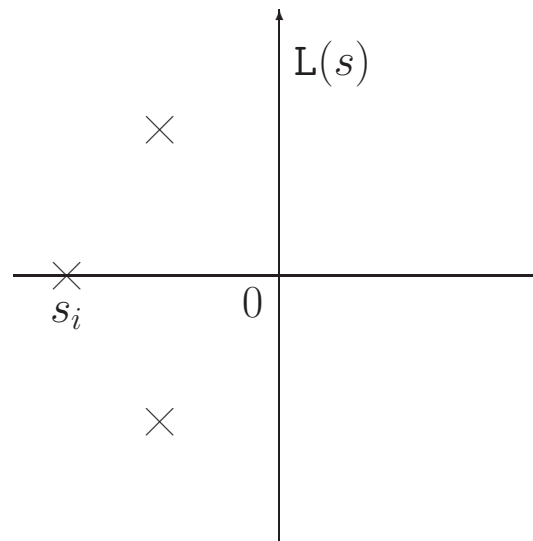
Minimum Phase System Revisited

11-8

Definition (Minimum-phase system) A system is called *minimum-phase* if both $L(\omega)$ and $1/L(\omega)$ are causal and stable.

Definition (Causal filter) A causal filter is one whose output depends only on past and present inputs.

Observation A system is minimum-phase if functions $L(s)$ and $1/L(s)$ are analytic in the right-hand plane $\text{Re}\{s\} > 0$.
(I.e., no poles and zeros satisfy $\text{Re}\{s\} > 0$.)



Implicitly, the above figure implies that $\mathbf{1}(\tau) \rightarrow 0$ as $\tau \rightarrow \infty$. See the two examples in the previous slide.

Minimum Phase System Revisited

11-9

$$\text{Example 11-3 } S(\omega) = \frac{25}{\omega^4 + 1}.$$

$$\text{Solution: } S(s) = \frac{25}{s^4 + 1} = \frac{25}{(s^2 + \sqrt{2}s + 1)(s^2 - \sqrt{2}s + 1)}$$

$$\Rightarrow L(s) = \frac{5}{s^2 + \sqrt{2}s + 1} \quad \left(S(s) = L(s)L(-s) \right)$$

$$\Rightarrow L(\omega) = \frac{5}{-\omega^2 + j\sqrt{2}\omega + 1}$$

$$\left(\Rightarrow |L(\omega)|^2 = \left| \frac{5}{-\omega^2 + j\sqrt{2}\omega + 1} \right|^2 = \frac{25}{(1 - \omega^2)^2 + 2\omega^2} = S(\omega) \right)$$

$$\begin{aligned} \Rightarrow l(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} L(\omega) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{5}{1 - \omega^2 + j\sqrt{2}\omega} e^{j\omega\tau} d\omega \\ &= \begin{cases} 5\sqrt{2} \sin(\tau/\sqrt{2}) e^{-\tau/\sqrt{2}}, & \tau > 0 \\ 0, & \tau < 0 \end{cases} \end{aligned}$$

□

Minimum-Phase Discrete-Time Processes

11-10

Concern (for discrete-time processes)

- How to represent a real **discrete** WSS process $\mathbf{x}[t]$ as a response of a **discrete** *minimum-phase* system $L[e^{j\omega}]$ with a **discrete** white input $\mathbf{i}[t]$ of unit power?

Definition (Minimum-phase system) A discrete system is called *minimum-phase* if both $L[e^{j\omega}]$ and $1/L[e^{j\omega}]$ are causal and stable.

(A system is stable if a bounded input (BI) always induces a bounded output (BO). As a result, a linear system is stable in the BIBO sense if all poles of the system are inside the unit circle in the z -plane.)

Definition (Causal filter) A causal filter is one whose output depends only on past and present inputs.

- A (discrete) process that can be represented as a response of a *minimum-phase* system $L[e^{j\omega}]$ with a white input $\mathbf{i}[t]$ of unit power is called *regular*.
- A formal definition of (discrete) regular processes is given below.

Definition (Discrete regular processes) A process $\mathbf{x}[t]$ is regular if

$$S_{xx}[\omega] = |L[e^{j\omega}]|^2$$

where $L[z]$ ($z = e^{j\omega}$) is analytic for $|z| > 1$.

- Roughly speaking, a function is **analytic** if its function values are determinate and finite (never indeterminate or infinity).

Some observations about $\mathbf{x}[t]$ so defined

- $R_{ii}[\tau] = \delta[\tau] \Rightarrow S_{ii}[\omega] = 1$, where $\delta[\tau] = \begin{cases} 1, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$ is the Kronecker delta function.
- $S_{xx}[\omega] = |\mathbf{L}[e^{j\omega}]|^2 S_{ii}[\omega] = |\mathbf{L}[e^{j\omega}]|^2$.
- So,

$$\mathbf{x}[t] = \sum_{\tau=-\infty}^{\infty} \mathbf{1}[\tau] \mathbf{i}[t - \tau],$$

where $\mathbf{L}[e^{j\omega}]$ is minimum-phase, determined in terms of the desired *real, positive, even, finite-area* $S_{xx}[\omega]$, and $\mathbf{1}[\tau] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{L}[e^{j\omega}] e^{j\omega\tau} d\omega$.

Conveniently, we will sometimes write $\mathbf{L}[e^{j\omega}]$ as $\mathbf{L}[\omega]$. These two expressions are actually equivalent.

Terminologies and Paley-Wiener Condition

11-12

Innovation: $\mathbf{i}[t]$ is called the *innovations* of $\mathbf{x}[t]$.

Innovation Filter: $L[\omega]$ is called the *innovation filter* of $\mathbf{x}[t]$.

Whitening Filter: $1/L[\omega]$ is called the *whitening filter* of $\mathbf{x}[t]$.

Lemma (Paley-Wiener condition) A process $\mathbf{x}[t]$ is regular if the Paley-Wiener condition holds, i.e.,

$$\int_{-\pi}^{\pi} |\log S_{xx}[\omega]| d\omega < \infty.$$

If $S_{xx}[\omega]$ is an integrable function, then the above condition reduces to

$$\int_{-\pi}^{\pi} \log(S_{xx}[\omega]) d\omega > -\infty.$$

- Obviously, $\int_{-\pi}^{\pi} |\log S_{xx}[\omega]| d\omega < \infty$ implies $\int_{-\pi}^{\pi} \log(S_{xx}[\omega]) d\omega > -\infty$.

We need to prove the converse is also true if $S_{xx}[\omega]$ integrable.

Terminologies and Paley-Wiener Condition

11-13

Claim $\int_{-\pi}^{\pi} |S[\omega]| d\omega < \infty$ and $\int_{-\pi}^{\pi} \log(S[\omega]) d\omega > -\infty \Rightarrow \int_{-\pi}^{\pi} |\log S[\omega]| d\omega < \infty$.

Proof: By

$$\int_{\{\omega \in [-\pi, \pi) : S[\omega] < 1\}} \log(S[\omega]) d\omega = \int_{-\pi}^{\pi} \log(S[\omega]) d\omega - \int_{\{\omega \in [-\pi, \pi) : S[\omega] \geq 1\}} \log(S[\omega]) d\omega,$$

we derive:

$$\begin{aligned} \int_{-\pi}^{\pi} |\log(S[\omega])| d\omega &= \int_{\{\omega \in [-\pi, \pi) : S[\omega] < 1\}} |\log(S[\omega])| d\omega + \int_{\{\omega \in [-\pi, \pi) : S[\omega] \geq 1\}} |\log(S[\omega])| d\omega \\ &= - \int_{\{\omega \in [-\pi, \pi) : S[\omega] < 1\}} \log(S[\omega]) d\omega + \int_{\{\omega \in [-\pi, \pi) : S[\omega] \geq 1\}} \log(S[\omega]) d\omega \\ &= - \int_{-\pi}^{\pi} \log(S[\omega]) d\omega + 2 \int_{\{\omega \in [-\pi, \pi) : S[\omega] \geq 1\}} \log(S[\omega]) d\omega \\ &\leq - \int_{-\pi}^{\pi} \log(S[\omega]) d\omega + 2 \int_{\{\omega \in [-\pi, \pi) : S[\omega] \geq 1\}} (S[\omega] - 1) d\omega \\ &\leq - \int_{-\pi}^{\pi} \log(S[\omega]) d\omega + 2 \int_{-\pi}^{\pi} (|S[\omega]| + 1) d\omega < \infty. \end{aligned}$$

□

Theorem (Page 424 in textbook: Chapter 9) There exists a unique function

$$H[z] = \sum_{k=0}^{\infty} h[k]z^{-k} \text{ for } h[0] > 0 \text{ and } |z| > 1$$

that is analytic together with its inverse in $|z| > 1$ satisfying

$$\sum_{k=0}^{\infty} |h[k]|^2 < \infty \quad \text{and} \quad S[\omega] = |H[e^{-j\omega}]|^2 \text{ a.e.,}$$

if, and only if, $S[\omega]$ as well as $\log(S[\omega])$ are integrable functions over $[-\pi, \pi)$, where

$$H[e^{-j\omega}] = \lim_{r \downarrow 1} H[re^{-j\omega}]$$

is defined as the exterior radial limit of $H[z]$ on the unit circle.

Rational Spectra for Discrete-Time Processes

11-15

How to find $\mathbf{L}[\omega]$ such that $|\mathbf{L}[\omega]|^2 = S[\omega]$ for a real, positive, even, finite-area $S[\omega]$.

- Observation 1: $S[\omega] (= S[e^{j\omega}] = S[e^{-j\omega}]) = S[-\omega]$ implies that $S[\omega]$ is a function of $\cos(\omega) = (e^{j\omega} + e^{-j\omega})/2$.
- Observation 2: $\mathbf{L}[\omega]$ can be easily determined if $S[\omega]$ is a rational spectrum.

Definition (Rational spectra) A rational spectrum is the ratio of two polynomials in $\cos(\omega)$:

$$S[\omega] = \frac{A(\cos(\omega))}{B(\cos(\omega))},$$

where $A(x)$ and $B(x)$ are both polynomials of x .

- Let $z = e^{j\omega}$. Then, $S[z] = A((z + z^{-1})/2)/B((z + z^{-1})/2)$.
- Observe that if z_i is a root (zero or pole) of $S[z]$, $1/z_i$ is also a root of $S[z]$. Also, the roots of $S[z]$ are either real or complex conjugate.
- Then, the roots of $S[z]$ are symmetric with respect to the **unit circle**. So we can separate them into two groups: **Inside** group that consists of all roots with $|z| < 1$, and the **outside** group that consists of all roots with $|z| > 1$.
- Form $\mathbf{L}[z]$ by the ratio of two polynomials with the **inside** roots of $S[z]$.

Rational Spectra for Discrete-Time Processes

11-16

$$\textbf{Example 11-4 } S[\omega] = \frac{5 - 4 \cos(\omega)}{10 - 6 \cos(\omega)}.$$

$$\textbf{Solution: } S[z] = \frac{5 - 2(z + z^{-1})}{10 - 3(z + z^{-1})} = \frac{2(1 - (1/2)z^{-1})}{3(1 - (1/3)z^{-1})} \cdot \frac{2(1 - (1/2)z)}{3(1 - (1/3)z)}$$

$$\Rightarrow L[z] = \frac{2(1 - (1/2)z^{-1})}{3(1 - (1/3)z^{-1})} \quad \left(S[z] = L[z]L[1/z] \right)$$

$$\Rightarrow L[\omega] = \frac{2(1 - (1/2)e^{-j\omega})}{3(1 - (1/3)e^{-j\omega})}$$

$$\left(\Rightarrow |L[\omega]|^2 = \left| \frac{2(1 - (1/2)e^{-j\omega})}{3(1 - (1/3)e^{-j\omega})} \right|^2 = S[\omega] \right)$$

$$\begin{aligned} \mathbf{1}[\tau] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathbf{L}[\omega] e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{2(1 - (1/2)e^{-j\omega})}{3(1 - (1/3)e^{-j\omega})} e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 - \frac{1/3}{1 - (1/3)e^{-j\omega}} \right) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - 3^{-1} [1 + 3^{-1}e^{-j\omega} + 3^{-2}e^{-j2\omega} + \dots]) e^{j\omega\tau} d\omega \\ &= \begin{cases} 0, & \tau < 0 \\ 1 - 3^{-1}, & \tau = 0 \\ -3^{-(1+\tau)}, & \tau > 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \mathbf{1}_{\text{whitening}}[\tau] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{\mathbf{L}[\omega]} e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{3(1 - (1/3)e^{-j\omega})}{2(1 - (1/2)e^{-j\omega})} e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(1 + \frac{1/2}{1 - (1/2)e^{-j\omega}} \right) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + 2^{-1} [1 + 2^{-1}e^{-j\omega} + 2^{-2}e^{-j2\omega} + \dots]) e^{j\omega\tau} d\omega \\ &= \begin{cases} 0, & \tau < 0 \\ 1 + 2^{-1}, & \tau = 0 \\ 2^{-(1+\tau)}, & \tau > 0 \end{cases} \end{aligned}$$

□

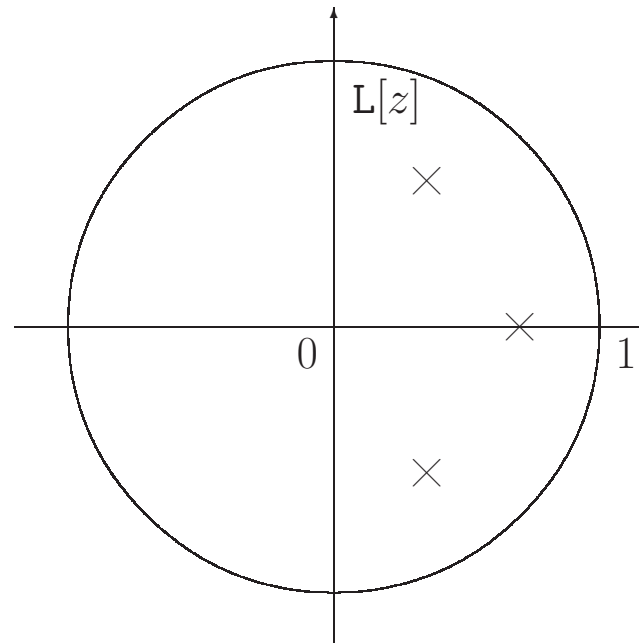
Rational Spectra for Discrete-Time Processes

11-19

Definition (Minimum-phase system) A system is called *minimum-phase* if both $L[\omega]$ and $1/L[\omega]$ are causal and stable.

Definition (Causal filter) A causal filter is one whose output depends only on past and present inputs.

Observation A discrete system is minimum-phase if functions $L[z]$ and $1/L[z]$ are analytic in the exterior $|z| > 1$ of the unit circle.



Properties Regarding Minimum-Phase

11-20

Filter with minimum group delay

- For all causal and stable systems that have the same magnitude response, the minimum phase system has the *minimum group delay*.
- Hence, a more appropriate name for *minimum-phase system* is the “*minimum group delay*” system.

Delay of a filter: What is a proper definition for filter delay?

$$\begin{array}{c} \mathbf{x}[t] \longrightarrow \boxed{\mathbf{L}[\omega] = e^{-j\omega n}} \longrightarrow \mathbf{y}[t] = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mathbf{X}[\omega] e^{-j\omega n}) e^{j\omega t} d\omega = \mathbf{x}[t - n] \end{array}$$

Hence, the delay of a filter can be “defined” as:

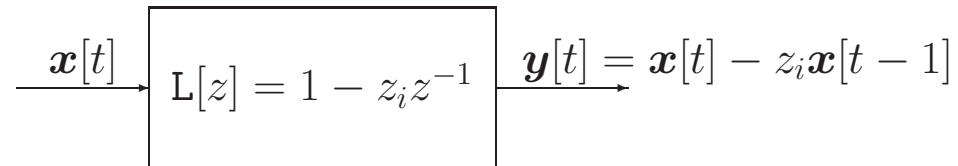
$$-\frac{d}{d\omega} (\arg\{\mathbf{L}[\omega]\}) = -\frac{d}{d\omega} (-\omega n) = n.$$

The filter is named *minimum phase* due to that it minimizes the “**phase change.**”

Properties Regarding Minimum-Phase

11-21

Example: Delay of a filter with a single zero



$$\begin{aligned} -\frac{d}{d\omega} \left(\arg\{1 - z_i e^{-j\omega}\} \right) &= -\frac{d}{d\omega} \left(\arg \left\{ |z_i|^{-1} - e^{-j(\omega - \arg\{z_i\})} \right\} \right) \\ &= -\frac{d}{d\omega} \left(\arg \left\{ |z_i|^{-1} - \cos(\omega - \arg\{z_i\}) + j \sin(\omega - \arg\{z_i\}) \right\} \right) \\ &= -\frac{d}{d\omega} \left(\tan^{-1} \left[\frac{\sin(\omega - \arg\{z_i\})}{|z_i|^{-1} - \cos(\omega - \arg\{z_i\})} \right] \right) \\ &= \frac{|z_i| - \cos(\omega - \arg\{z_i\})}{|z_i| + |z_i|^{-1} - 2 \cos(\omega - \arg\{z_i\})}. \end{aligned}$$

See the next Slide.

Apparently, the choice between $z_i = |z_i| e^{j \arg\{z_i\}}$ and $1/z_i^* = |z_i|^{-1} e^{j \arg\{z_i\}}$ (or z_i^* and $1/z_i$), which minimizes the *filter delay*, is the one lying in the interior of unit circle since

$$-\frac{d}{d\omega} \left(\arg\{1 - z_i e^{-j\omega}\} \right) = \frac{|z_i| + \text{fixed}}{\text{fixed}}.$$

Properties Regarding Minimum-Phase

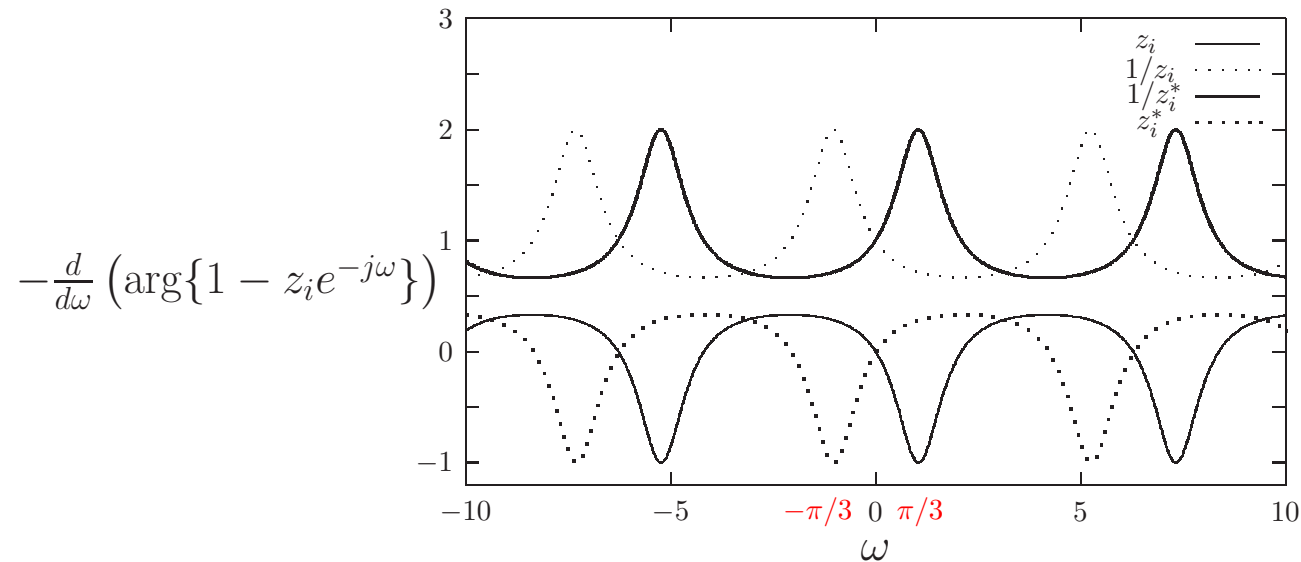
11-22

$$\begin{aligned} & \frac{d}{d\omega} \left(\tan^{-1} \left[\frac{\sin(\omega - \arg\{z_i\})}{|z_i|^{-1} - \cos(\omega - \arg\{z_i\})} \right] \right) && \boxed{\frac{d}{d\omega} \tan^{-1} \left(\frac{a}{b} \right) = \frac{ba' - ab'}{a^2 + b^2}} \\ &= \frac{[|z_i|^{-1} - \cos(\omega - \arg\{z_i\})] \left(\frac{d}{d\omega} [\sin(\omega - \arg\{z_i\})] \right)}{\sin^2(\omega - \arg\{z_i\}) + [|z_i|^{-1} - \cos(\omega - \arg\{z_i\})]^2} \\ & \quad - \frac{\sin(\omega - \arg\{z_i\}) \left(\frac{d}{d\omega} [|z_i|^{-1} - \cos(\omega - \arg\{z_i\})] \right)}{\sin^2(\omega - \arg\{z_i\}) + [|z_i|^{-1} - \cos(\omega - \arg\{z_i\})]^2} \\ &= \frac{[|z_i|^{-1} - \cos(\omega - \arg\{z_i\})] \cos(\omega - \arg\{z_i\}) - \sin^2(\omega - \arg\{z_i\})}{1 - 2|z_i|^{-1} \cos(\omega - \arg\{z_i\}) + |z_i|^{-2}} \\ &= \frac{|z_i|^{-1} \cos(\omega - \arg\{z_i\}) - 1}{1 - 2|z_i|^{-1} \cos(\omega - \arg\{z_i\}) + |z_i|^{-2}} \\ &= \frac{\cos(\omega - \arg\{z_i\}) - |z_i|}{|z_i| - 2 \cos(\omega - \arg\{z_i\}) + |z_i|^{-1}} \end{aligned}$$

Properties Regarding Minimum-Phase

11-23

Example. Take $|z_i| = \frac{1}{2}$ and $\arg\{z_i\} = \frac{\pi}{3}$.



- The figure shows that

$$\frac{|z_i| - \cos(\omega - \arg\{z_i\})}{|z_i| + |z_i|^{-1} - 2 \cos(\omega - \arg\{z_i\})} \quad \text{and} \quad \frac{|z_i^*| - \cos(\omega - \arg\{z_i^*\})}{|z_i^*| + |z_i^*|^{-1} - 2 \cos(\omega - \arg\{z_i^*\})}$$

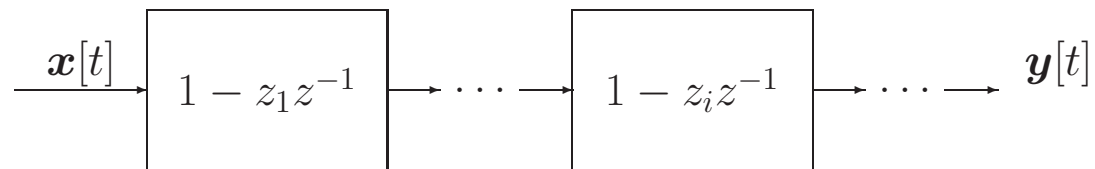
are identical except with some shift.

- The figure shows that the phase change corresponding to z_i is always smaller than that corresponding to $1/z_i$ (and of course, to $1/z_i^*$). So, the choice between $z_i = |z_i|e^{j \arg\{z_i\}}$ and $1/z_i = |z_i|^{-1}e^{-j \arg\{z_i\}}$, which minimizes the *filter delay*, is the one lying in the interior of unit circle.

Properties Regarding Minimum-Phase

11-24

Delay of a filter with multiple zeros $L[z] = \prod_i (1 - z_i z^{-1})$



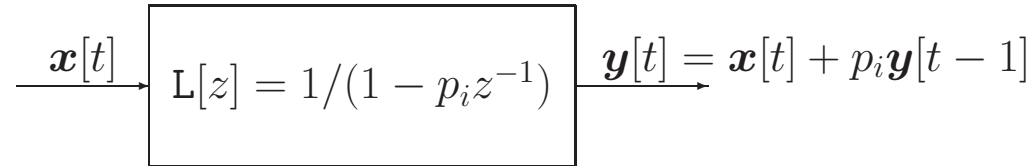
Choose half of the zeros, among all the zero-pairs (one in inside group and one in outside group) of the target $S[z] = L[z]L[1/z]$, such that the group delay is minimized.

Apparently, the choice of all zeros in the inside group will satisfy the need.

Properties Regarding Minimum-Phase

11-25

Delay of a filter with a single pole



$$L[e^{j\omega}] = \frac{1 - p_i^* e^{j\omega}}{|1 - p_i e^{-j\omega}|^2} \Rightarrow \arg \{L[e^{j\omega}]\} = \arg \{1 - p_i^* e^{j\omega}\}$$

and

$$-\frac{d}{d\omega} (\arg\{1 - p_i^* e^{j\omega}\}) = \frac{|p_i^*| - \cos(\omega + \arg\{p_i^*\})}{|p_i^*| + |p_i^*|^{-1} - 2 \cos(\omega + \arg\{p_i^*\})}.$$

Again, the choice between $p_i^* = |p_i^*| e^{j \arg\{p_i^*\}}$ and $1/p_i = |p_i^*|^{-1} e^{j \arg\{p_i^*\}}$ (or p_i and $1/p_i^*$), which minimizes the *filter delay*, is the one lying in the interior of unit circle since

$$-\frac{d}{d\omega} (\arg\{1 - p_i^* e^{j\omega}\}) = \frac{|p_i^*| + \text{fixed}}{\text{fixed}}.$$

All the conclusions for zeros can be applied to poles.

Properties Regarding Minimum-Phase

11-26

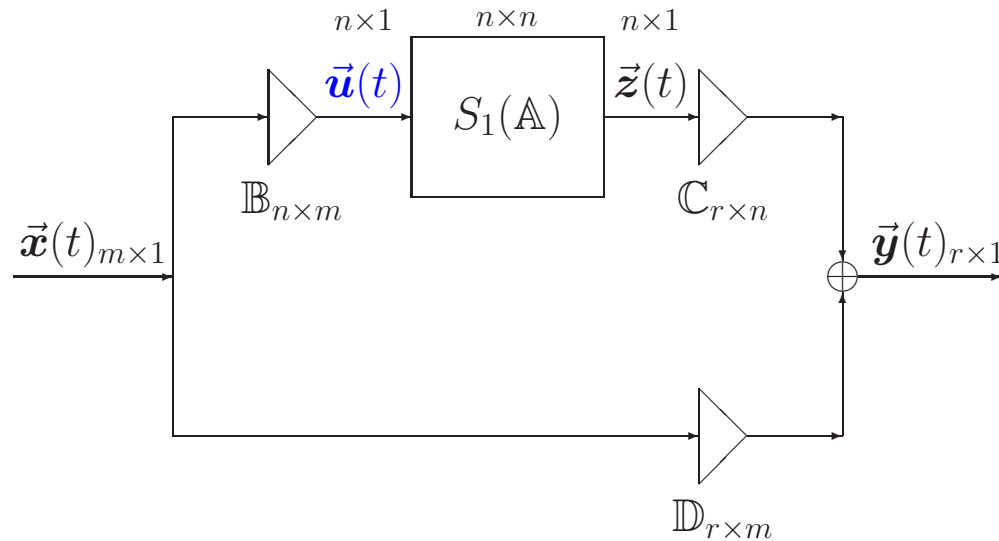
Delay of a filter with multiple zeros and multiple poles

$$L[z] = \frac{\prod_i (1 - z_i z^{-1})}{\prod_k (1 - p_k z^{-1})}$$

Choose half of the zeros and poles, among all the zero-pairs and pole-pairs (one in inside group and one in outside group) of the desired $S[z] = L[z]L[1/z]$ such that the group delay is minimized.

Apparently, the choice of all zeros and poles in the inside group will satisfy the need.

The end of Section 11-1 Factorization and Innovations



A system with state variables

- Consider a system with input $\vec{x}(t)$ and output $\vec{y}(t)$, in which their relationship is defined through an internal state variable $\vec{z}(t)$ as:

$$\begin{cases} \frac{d}{dt}\vec{z}(t) = \mathbb{A}\vec{z}(t) + \vec{u}(t) = \mathbb{A}\vec{z}(t) + \mathbb{B}\vec{x}(t) & (*) \\ \vec{y}(t) = \mathbb{C}\vec{z}(t) + \mathbb{D}\vec{x}(t) \end{cases}$$

The relationship between input $\vec{u}(t)$ and output $\vec{z}(t)$ of the subsystem S_1 is given by (*).

Terminology

- The order of the system is defined as the dimension of the state variable $\vec{z}(t)$, which is n in our case.

Derivation of the impulse response

- The impulse response of the subsystem S_1 can be derived from relationship

$$\vec{z}(t)_{n \times 1} = \int_{-\infty}^{\infty} \phi(\alpha)_{n \times n} \vec{u}(t - \alpha)_{n \times 1} d\alpha \quad \text{equivalently} \quad \vec{z}(s)_{n \times 1} = \phi(s)_{n \times n} \vec{u}(s)_{n \times 1}.$$

Taking the Laplace transform of both sides of Eq. (*) yields:

$$\begin{aligned} s\vec{z}(s)_{n \times 1} &= \mathbb{A}_{n \times n} \vec{z}(s)_{n \times 1} + \vec{u}(s)_{n \times 1} \\ \Rightarrow s\phi(s)_{n \times n} \vec{u}(s)_{n \times 1} &= \mathbb{A}_{n \times n} \phi(s)_{n \times n} \vec{u}(s)_{n \times 1} + \vec{u}(s)_{n \times 1} \\ \Rightarrow s\Phi(s)_{n \times n} - \mathbb{A}_{n \times n} \Phi(s)_{n \times n} &= \mathbb{I}_{n \times n} \\ \Rightarrow \Phi(s)_{n \times n} &= (s\mathbb{I}_{n \times n} - \mathbb{A}_{n \times n})^{-1} \\ \Rightarrow \phi(t)_{n \times n} &= \exp\{\mathbb{A}_{n \times n} t\} \quad t > 0. \end{aligned}$$

where \mathbb{I} is the identity matrix.

$$e^{\mathbb{A}t} \triangleq \begin{cases} \mathbb{S}e^{\Lambda t}\mathbb{S}^{-1} = \mathbb{S} \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{bmatrix} \mathbb{S}^{-1}, & \text{if } \mathbb{S}^{-1} \text{ exists} \\ \sum_{k=0}^{\infty} \frac{1}{k!} (\mathbb{A}t)^k, & \text{holds no matter whether} \\ & \mathbb{S}^{-1} \text{ exists or not} \end{cases}$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of \mathbb{A} , and \mathbb{S} is the matrix with its columns being the linearly independent eigenvectors of \mathbb{A} .

Derivation of the impulse response (continued)

- For the overall system,

$$\begin{aligned}
 \vec{y}(t)_{r \times 1} &= \mathbf{C}_{r \times n} \vec{z}(t)_{n \times 1} + \mathbf{D}_{r \times m} \vec{x}(t)_{m \times 1} \\
 &= \int_{-\infty}^{\infty} \mathbf{C}_{r \times n} \phi(\alpha)_{n \times n} \vec{u}(t - \alpha)_{n \times 1} d\alpha + \int_{-\infty}^{\infty} \delta(\alpha) \mathbf{D}_{r \times m} \vec{x}(t - \alpha)_{m \times 1} d\alpha \\
 &= \int_{-\infty}^{\infty} \mathbf{C}_{r \times n} \phi(\alpha)_{n \times n} \mathbf{B}_{n \times m} \vec{x}(t - \alpha)_{m \times 1} d\alpha + \int_{-\infty}^{\infty} \delta(\alpha) \mathbf{D}_{r \times m} \vec{x}(t - \alpha)_{m \times 1} d\alpha \\
 &= \int_{-\infty}^{\infty} (\mathbf{C}_{r \times n} \phi(\alpha)_{n \times n} \mathbf{B}_{n \times m} + \delta(\alpha) \mathbf{D}_{r \times m}) \vec{x}(t - \alpha)_{m \times 1} d\alpha.
 \end{aligned}$$

Hence,

$$h(t)_{r \times m} = \mathbf{C}_{r \times n} \phi(t)_{n \times n} \mathbf{B}_{n \times m} + \delta(t) \mathbf{D}_{r \times m}$$

and

$$H(s)_{r \times m} = \mathbf{C}_{r \times n} \Phi(s)_{n \times n} \mathbf{B}_{n \times m} + \mathbf{D}_{r \times m} = \boxed{\mathbf{C}_{r \times n} (s\mathbf{I}_{n \times n} - \mathbf{A}_{n \times n})^{-1} \mathbf{B}_{n \times m} + \mathbf{D}_{r \times m}}.$$

By Theorem 9-4 that tells:

$$\begin{cases} S_{xy}(\omega)_{m \times r} = S_{xx}(\omega)_{m \times m} H^\dagger(\omega)_{r \times m} \\ S_{yy}(\omega)_{r \times r} = H(\omega)_{r \times m} S_{xy}(\omega)_{m \times r} \\ \quad \quad \quad = H(\omega)_{r \times m} S_{xx}(\omega)_{m \times m} H^\dagger(\omega)_{r \times m} \end{cases}$$

we can infer that,

$$\begin{cases} S_{xy}(s)_{m \times r} = S_{xx}(s)_{m \times m} H^\dagger(-s)_{r \times m} \\ S_{yy}(s)_{r \times r} = H(s)_{r \times m} S_{xy}(s)_{m \times r} \\ \quad \quad \quad = H(s)_{r \times m} S_{xx}(s)_{m \times m} H^\dagger(-s)_{r \times m} \end{cases}$$

Example 1

- Suppose $r = n = m$ and $\mathbb{B}_{n \times n} = \mathbb{C}_{n \times n} = \mathbb{I}_{n \times n}$ and $\mathbb{D}_{n \times n} = \mathbf{0}_{n \times n}$. Then,

$$\begin{cases} \frac{d}{dt} \vec{z}(t) = \mathbb{A} \vec{z}(t) + \vec{u}(t) = \mathbb{A} \vec{z}(t) + \vec{x}(t) \\ \vec{y}(t) = \vec{z}(t) \end{cases}$$

implies

$$\frac{d}{dt} \vec{y}(t) = \mathbb{A} \vec{y}(t) + \vec{x}(t).$$

- Then,

$$H(s)_{n \times n} = \mathbb{C}_{r \times n} (s\mathbb{I}_{n \times n} - \mathbb{A}_{n \times n})^{-1} \mathbb{B}_{n \times m} + \mathbb{D}_{r \times m} = \boxed{(s\mathbb{I}_{n \times n} - \mathbb{A}_{n \times n})^{-1}}.$$

Example 2

- Suppose

$$\mathbf{y}^{(n)}(t) + a_1 \mathbf{y}^{(n-1)}(t) + \cdots + a_n \mathbf{y}(t) = \mathbf{x}(t).$$

- By assuming that $\vec{\mathbf{z}}(t) = [\mathbf{y}(t), \mathbf{y}^{(1)}(t), \dots, \mathbf{y}^{(n-1)}(t)]^T$, the system can be equivalently transformed to:

$$\left\{ \begin{array}{l} \frac{d}{dt} \vec{\mathbf{z}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}}_{\mathbb{A}} \vec{\mathbf{z}}(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbb{B}} \mathbf{x}(t) \\ \\ \mathbf{y}(t) = \underbrace{[1 \ 0 \ \cdots \ 0]}_{\mathbb{C}} \vec{\mathbf{z}}(t) + \underbrace{0}_{\mathbb{D}} \end{array} \right.$$

Finite-Order Systems and State Variables

11-34

- Hence,

$$\begin{aligned} H(s)_{r \times m} &= \boxed{\mathbb{C}_{r \times n} (s\mathbb{I}_{n \times n} - \mathbb{A}_{n \times n})^{-1} \mathbb{B}_{n \times m} + \mathbb{D}_{r \times m}} \\ &= \mathbb{C}_{1 \times n} (s\mathbb{I}_{n \times n} - \mathbb{A}_{n \times n})^{-1} \mathbb{B}_{n \times 1} \\ &= [1 \ 0 \ \cdots \ 0] \begin{bmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ 0 & 0 & s & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & s + a_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n}. \end{aligned}$$

□

Definition (Finite-order processes) A (WSS) process $\mathbf{x}(t)$ is of finite order if its innovation filter is a rational function of s , i.e.,

$$L(s) = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{s^n + a_1 s^{n-1} + \dots + a_n} = \frac{N(s)}{D(s)},$$

satisfying that $N(s)$ and $D(s)$ are Hurwitz polynomials.

A Hurwitz polynomial is a polynomial whose zeros are located in the left half-plane of the complex plane, namely, the real part of every zero is negative.

Autocorrelation function of finite-order process $\mathbf{x}(t)$

- Let $\{s_i\}_{i=1}^n$ be the roots of $D(s)$, and assume $m < n$.
- Then, $L(s)$ can be expanded into partial fractions as:

$$L(s) = \sum_{i=1}^n \frac{\gamma_i}{s - s_i} \quad \text{and} \quad \mathbf{l}(\tau) = \sum_{i=1}^n \gamma_i e^{s_i \tau} \int_{-\infty}^{\tau} \delta(u) du.$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\gamma_i}{j\omega - s_i} e^{j\omega\tau} d\omega = \gamma_i e^{s_i\tau} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{j\omega'} e^{j\omega'\tau} d\omega' \right) \text{ for } j\omega' = j\omega - s_i.$$

See Slide 11-6.

Finite-Order Processes

11-36

- We can then derive:

$$\begin{aligned} S_{xx}(s) &= \mathbf{L}(s)\mathbf{L}(-s) \\ &= \left(\sum_{i=1}^n \frac{\gamma_i}{s - s_i} \right) \left(\sum_{k=1}^n \frac{\gamma_k}{-s - s_k} \right) \\ &= \sum_{i=1}^n \sum_{k=1}^n \frac{\gamma_i \gamma_k}{(s - s_i)(-s - s_k)} \\ &= \sum_{i=1}^n \sum_{k=1}^n \left(\frac{-\gamma_i \gamma_k / (s_i + s_k)}{s - s_i} + \frac{-\gamma_i \gamma_k / (s_i + s_k)}{-s - s_k} \right) \\ &= \sum_{i=1}^n \sum_{k=1}^n \frac{-\gamma_i \gamma_k / (s_i + s_k)}{s - s_i} + \sum_{i=1}^n \sum_{k=1}^n \frac{-\gamma_i \gamma_k / (s_i + s_k)}{-s - s_k} \\ &= \sum_{i=1}^n \frac{\alpha_i}{s - s_i} + \sum_{k=1}^n \frac{\alpha_k}{-s - s_k} \triangleq S_{xx}^+(s) + S_{xx}^+(-s), \end{aligned}$$

where

$$\alpha_k = \gamma_k \sum_{i=1}^n \frac{\gamma_i}{-s_k - s_i} = \gamma_k \mathbf{L}(-s_k).$$

Finite-Order Processes

11-37

- This gives that:

$$\begin{aligned} R_{xx}^+(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}^+(\omega) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{i=1}^n \frac{\alpha_i}{j\omega - s_i} \right) e^{j\omega\tau} d\omega \\ &= \begin{cases} \sum_{i=1}^n \alpha_i e^{s_i\tau}, & \tau > 0 \\ 0, & \tau < 0 \end{cases} \end{aligned}$$

and

$$\begin{aligned} R_{xx}(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega) e^{j\omega\tau} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (S_{xx}^+(\omega) + S_{xx}^+(-\omega)) e^{j\omega\tau} d\omega \\ &= R_{xx}^+(\tau) + R_{xx}^+(-\tau) \\ &= R_{xx}^+(|\tau|) \quad (\text{for } \tau \neq 0) \end{aligned}$$

Finite-Order Processes

11-38

Example 11-5 $L(s) = 1/(s + \alpha)$

Solution:

$$S_{xx}(s) = \frac{1}{(s + \alpha)(-s + \alpha)} = \frac{1/(2\alpha)}{s + \alpha} + \frac{1/(2\alpha)}{-s + \alpha}.$$

Then,

$$R_{xx}(\tau) = \frac{1}{2\alpha} e^{-\alpha|\tau|}.$$

□

Example 11-6 $x''(t) + 3x'(t) + 2x(t) = i(t).$

Solution

$$L(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{s + 1} + \frac{-1}{s + 2}$$

$$\Rightarrow S_{xx}(s) = L(s)L(-s) = \frac{1}{(s^2 + 3s + 2)(s^2 - 3s + 2)} = \frac{s/12 + 1/4}{s^2 + 3s + 2} + \frac{-s/12 + 1/4}{s^2 - 3s + 2}.$$

Hence,

$$S_{xx}^+(s) = \frac{1/6}{s + 1} + \frac{(-1/12)}{s + 2} \Rightarrow R_{xx}(\tau) = R_{xx}^+(|\tau|) = \frac{1}{6} e^{-|\tau|} - \frac{1}{12} e^{-2|\tau|}.$$

□

Given

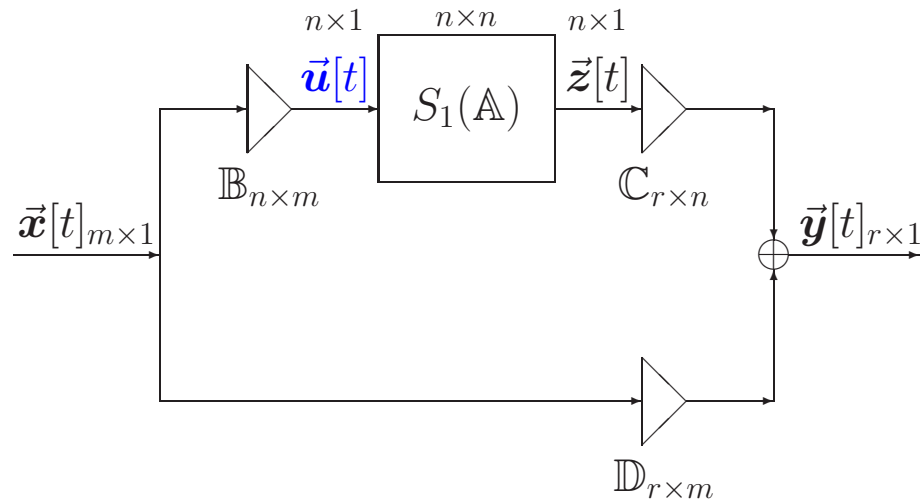
$$\mathbf{x}(t) = \int_{-\infty}^{\infty} \mathbf{1}(\tau) \mathbf{i}(t - \tau) d\tau,$$

we derive

$$\begin{aligned} R_{xx}(\tau) &= E[\mathbf{x}(t + \tau) \mathbf{x}(t)] \\ &= E \left[\left(\int_{-\infty}^{\infty} \mathbf{1}(u) \mathbf{i}(t + \tau - u) du \right) \left(\int_{-\infty}^{\infty} \mathbf{1}(v) \mathbf{i}(t - v) dv \right) \right] \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{1}(u) \mathbf{1}(v) E[\mathbf{i}(t + \tau - u) \mathbf{i}(t - v)] dudv \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{1}(u) \mathbf{1}(v) \delta(\tau - u + v) dudv \\ &= \int_{-\infty}^{\infty} \mathbf{1}(v) \mathbf{1}(\tau + v) dv \quad \left(= \int_{-\infty}^{\infty} \mathbf{1}(-v) \mathbf{1}(\tau - v) dv = \mathbf{1}(-\tau) * \mathbf{1}(\tau) \right) \end{aligned}$$

Thus,

$$\begin{aligned} R_{xx}(-\tau) &= \int_{-\infty}^{\infty} \mathbf{1}(v) \mathbf{1}(-\tau + v) dv \\ &= \int_{-\infty}^{\infty} \mathbf{1}(u + \tau) \mathbf{1}(u) du \quad (u = -\tau + v) = R_{xx}(\tau) \end{aligned}$$



A system with state variables

- Consider a system with input $\vec{x}[t]$ and output $\vec{y}[t]$, in which their relationship is defined through an internal state variable $\vec{z}[t]$ as:

$$\begin{cases} \vec{z}[t+1] = \mathbb{A}\vec{z}[t] + \vec{u}[t] = \mathbb{A}\vec{z}[t] + \mathbb{B}\vec{x}[t] & (*) \\ \vec{y}[t] = \mathbb{C}\vec{z}[t] + \mathbb{D}\vec{x}[t] \end{cases}$$

The relationship between input $\vec{u}[t]$ and output $\vec{z}[t]$ of the subsystem S_1 is given by (*).

Terminology

- The order of the system is defined as the dimension of the state variable $\vec{z}[t]$, which is n in our case.

Derivation of the impulse response

- The impulse response of the subsystem S_1 can be derived from relationship

$$\vec{z}[t]_{n \times 1} = \sum_{\alpha=-\infty}^{\infty} \phi[\alpha]_{n \times n} \vec{u}[t - \alpha]_{n \times 1} \text{ equivalently } \vec{z}[z]_{n \times 1} = \phi[z]_{n \times n} \vec{u}[z]_{n \times 1}$$

Taking the z -transform of both sides of (*) yields:

$$\begin{aligned} z\vec{z}[z]_{n \times 1} &= \mathbb{A}_{n \times n} \vec{z}[z]_{n \times 1} + \vec{u}[z]_{n \times 1} \\ \Rightarrow z\phi[z]_{n \times n} \vec{u}[z]_{n \times 1} &= \mathbb{A}_{n \times n} \phi[z]_{n \times n} \vec{u}[z]_{n \times 1} + \vec{u}[z]_{n \times 1} \\ \Rightarrow z\Phi[z]_{n \times n} - \mathbb{A}_{n \times n} \Phi[z]_{n \times n} &= \mathbb{I}_{n \times n} \\ \Rightarrow \Phi[z]_{n \times n} &= (z\mathbb{I}_{n \times n} - \mathbb{A}_{n \times n})^{-1} \\ \Rightarrow \phi[t]_{n \times n} &= \exp \{ \mathbb{A}_{n \times n} t \}. \end{aligned}$$

where \mathbb{I} is the identity matrix.

Derivation of the impulse response (continued)

- For the overall system,

$$\begin{aligned}
 \vec{y}[t]_{r \times 1} &= \mathbb{C}_{r \times n} \vec{z}[t]_{n \times 1} + \mathbb{D}_{r \times m} \vec{x}[t]_{m \times 1} \\
 &= \sum_{\alpha=-\infty}^{\infty} \mathbb{C}_{r \times n} \phi[\alpha]_{n \times n} \vec{u}[t - \alpha]_{n \times 1} + \sum_{\alpha=-\infty}^{\infty} \delta[\alpha] \mathbb{D}_{r \times m} \vec{x}[t - \alpha]_{m \times 1} \\
 &= \sum_{\alpha=-\infty}^{\infty} \mathbb{C}_{r \times n} \phi[\alpha]_{n \times n} \mathbb{B}_{n \times m} \vec{x}[t - \alpha]_{m \times 1} + \sum_{\alpha=-\infty}^{\infty} \delta[\alpha] \mathbb{D}_{r \times m} \vec{x}[t - \alpha]_{m \times 1} \\
 &= \sum_{\alpha=-\infty}^{\infty} (\mathbb{C}_{r \times n} \phi[\alpha]_{n \times n} \mathbb{B}_{n \times m} + \delta[\alpha] \mathbb{D}_{r \times m}) \vec{x}[t - \alpha]_{m \times 1}.
 \end{aligned}$$

Hence,

$$h[t]_{r \times m} = \mathbb{C}_{r \times n} \phi[t]_{n \times n} \mathbb{B}_{n \times m} + \delta[t] \mathbb{D}_{r \times m}$$

and

$$H[z]_{r \times m} = \mathbb{C}_{r \times n} \Phi[z]_{n \times n} \mathbb{B}_{n \times m} + \mathbb{D}_{r \times m} = \boxed{\mathbb{C}_{r \times n} (z\mathbb{I}_{n \times n} - \mathbb{A}_{n \times n})^{-1} \mathbb{B}_{n \times m} + \mathbb{D}_{r \times m}}.$$

Definition (Discrete finite-order processes) A (WSS) discrete process $\mathbf{x}[t]$ is of finite order if its innovation filter is a rational function of z , i.e.,

$$\mathbf{L}[z] = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_n z^{-n}} = \frac{N[z]}{D[z]},$$

satisfying that the roots of $N[z]$ and $D[z]$ are within the unit circle.

Autocorrelation function of discrete finite-order process $\mathbf{x}[t]$

- Let $\{z_i\}_{i=1}^n$ be the roots of $D[z]$, and assume $m \leq n$.
 - We allow $m = n$ with $z_1 = 0$ in some practical case. In such case, $\frac{\gamma_1}{1-z_1 z^{-1}}$ below is equal to $\gamma_1 = b_n/a_n$.
 - Here, we further assume that $z_i \neq 0$ for $i \geq 2$.
- Then, $\mathbf{L}[z]$ can be expanded into partial fractions as:

$$\mathbf{L}[z] = \sum_{i=1}^n \frac{\gamma_i}{1 - z_i z^{-1}} \text{ and } \mathbf{l}(\tau) = \gamma_1 \delta[\tau] \mathbf{1}\{z_1 = 0\} + \gamma_1 z_1^\tau \mathbf{1}\{\tau \geq 0\} \mathbf{1}\{z_1 \neq 0\} + \sum_{i=2}^n \gamma_i z_i^\tau \mathbf{1}\{\tau \geq 0\}.$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\gamma_i}{1 - z_i e^{-j\omega}} e^{j\omega\tau} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \gamma_i (1 + z_i e^{-j\omega} + z_i^2 e^{-j2\omega} + \dots) e^{j\omega\tau} d\omega.$$

Discrete Finite-Order Processes

11-44

- We can then derive (and correct (11-37) in text) that:

$$\begin{aligned}
 S_{xx}[z] &= \mathbf{L}[z]\mathbf{L}[z^{-1}] \\
 &= \left(\sum_{i=1}^n \frac{\gamma_i}{1 - z_i z^{-1}} \right) \left(\sum_{k=1}^n \frac{\gamma_k}{1 - z_k z} \right) \\
 &= \sum_{i=1}^n \sum_{k=1}^n \frac{\gamma_i \gamma_k}{(1 - z_i z^{-1})(1 - z_k z)} \\
 &= \sum_{i=1}^n \sum_{k=1}^n \left(\frac{\gamma_i \gamma_k / (1 - z_i z_k)}{1 - z_i z^{-1}} + \frac{\gamma_i \gamma_k / (1 - z_i z_k)}{1 - z_k z} - \frac{\gamma_i \gamma_k}{1 - z_i z_k} \right) \\
 &= \sum_{i=1}^n \sum_{k=1}^n \frac{\gamma_i \gamma_k / (1 - z_i z_k)}{1 - z_i z^{-1}} + \sum_{i=1}^n \sum_{k=1}^n \frac{\gamma_i \gamma_k / (1 - z_i z_k)}{1 - z_k z} - \sum_{i=1}^n \sum_{k=1}^n \frac{\gamma_i \gamma_k}{1 - z_i z_k} \\
 &= \sum_{i=1}^n \frac{\alpha_i}{1 - z_i z^{-1}} + \sum_{k=1}^n \frac{\alpha_k}{1 - z_k z} - \sum_{i=1}^n \alpha_i = S_{xx}^+[z] + S_{xx}^+[1/z] - \sum_{i=1}^n \alpha_i,
 \end{aligned}$$

where

$$\alpha_k = \gamma_k \sum_{i=1}^n \frac{\gamma_i}{1 - z_i z_k} = \gamma_k \mathbf{L}[z_k^{-1}].$$

- This gives that:

$$\begin{aligned}
 R_{xx}^+[\tau] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}^+[e^{j\omega}] e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{i=1}^n \frac{\alpha_i}{1 - z_i e^{-j\omega}} \right) e^{j\omega\tau} d\omega \\
 &= \begin{cases} \alpha_1 \delta[\tau] \mathbf{1}\{z_1 = 0\} + \alpha_1 z_1^\tau \mathbf{1}\{z_1 \neq 0\} + \sum_{i=2}^n \alpha_i z_i^\tau, & \tau \geq 0 \\ 0, & \tau < 0 \end{cases}
 \end{aligned}$$

and

$$\begin{aligned}
 R_{xx}[\tau] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{xx}[e^{j\omega}] e^{j\omega\tau} d\omega \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(S_{xx}^+[e^{j\omega}] + S_{xx}^+[e^{-j\omega}] - \sum_{i=1}^n \alpha_i \right) e^{j\omega\tau} d\omega \\
 &= R_{xx}^+[\tau] + R_{xx}^+[-\tau] - \delta[\tau] R_{xx}^+[0] \\
 &= R_{xx}^+[|\tau|].
 \end{aligned}$$

Autoregressive Processes

11-46

Definition (AR processes) The discrete (finite-order) process $\mathbf{x}[t]$ is called autoregressive (AR) if its innovation filter is of the form:

$$L[z] = \frac{b_0}{1 + a_1 z^{-1} + \dots + a_n z^{-n}}.$$

Remarks

- For AR processes,

$$\mathbf{x}[t] + a_1 \mathbf{x}[t - 1] + \dots + a_n \mathbf{x}[t - n] = b_0 \mathbf{i}[t]. \quad (11.1)$$

- It is named AR because the output will continue indefinitely in a self-regressive fashion only with one excitation.
- Since $\mathbf{x}[t - m]$ can be completely determined by

$$\mathbf{x}[t - m - 1] \text{ upto } \mathbf{x}[t - m - n] \text{ and } \mathbf{i}[t - m],$$

it only depends on

$$\mathbf{i}[t - m], \mathbf{i}[t - m - 1], \mathbf{i}[t - m - 2], \dots$$

Accordingly **under the assumption that $\mathbf{x}[t]$ is WSS,**

$$R_{xi}[-m] = E\{\mathbf{x}[t - m]\mathbf{i}[t]\} = E\{\mathbf{x}[t - m]\}E\{\mathbf{i}[t]\} = 0 \text{ for } m > 0.$$

Autoregressive Processes

- By multiplying $\mathbf{i}[t]$ followed by taking expectation of both sides of (11.1), we obtain:

$$R_{xi}[0] + a_1 R_{xi}[-1] + a_2 R_{xi}[-2] + \cdots + a_n R_{xi}[-n] = R_{xi}[0] = b_0.$$

- By multiplying $\mathbf{x}[t - m]$ for $0 \leq m \leq n$ followed by taking expectation of both sides of (11.1), we obtain:

$$\begin{aligned} \times \mathbf{x}[t] & : R_{xx}[0] + a_1 R_{xx}[-1] + \cdots + a_n R_{xx}[-n] = b_0^2 \\ \times \mathbf{x}[t - 1] & : R_{xx}[1] + a_1 R_{xx}[0] + \cdots + a_n R_{xx}[-n + 1] = 0 \\ & \vdots \\ \times \mathbf{x}[t - n] & : R_{xx}[n] + a_1 R_{xx}[n - 1] + \cdots + a_n R_{xx}[0] = 0, \end{aligned}$$

or equivalently,

$$\begin{bmatrix} R_{xx}[0] & R_{xx}[-1] & R_{xx}[-2] & \cdots & R_{xx}[-n] \\ R_{xx}[1] & R_{xx}[0] & R_{xx}[-1] & \cdots & R_{xx}[-n + 1] \\ R_{xx}[2] & R_{xx}[1] & R_{xx}[0] & \cdots & R_{xx}[-n + 2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{xx}[n] & R_{xx}[n - 1] & R_{xx}[n - 2] & \cdots & R_{xx}[0] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_0^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is named the *Yule-Walker* equations.

The Yule-Walker equations can be used to determine a_1, \cdots, a_n and b_0 for known $R_{xx}[m]$, or to determine $R_{xx}[m]$ recursively for known a_1, \cdots, a_n and b_0 .

Autoregressive Processes

11-48

Example 11-7 $x[t] - ax[t - 1] = bi[t]$.

Solution:

- $L[z] = \frac{b}{1 - az^{-1}} \Rightarrow z_1 = a$ and $\gamma_1 = b$ and $\alpha_1 = \gamma_1 L[1/z_1] = b^2/(1 - a^2)$.

- Then, $R_{xx}[\tau] = \alpha_1 z_1^{|\tau|} = \frac{b^2}{1 - a^2} a^{|\tau|}$. □

If $a > 1$, then $b(1 + az^{-1} + a^2z^{-2} + \dots)$ does not converge unless $|z| < 1/|a|$; hence,

$$\frac{b}{1 - az^{-1}} = b(1 + az^{-1} + a^2z^{-2} + \dots)$$

is not valid for $|z| = |e^{j\omega}| = 1$. In short, an AR process with roots outside the unit circle is not stationary!

Two cases that are not included in Slide 11-43:

1. Case of $m = 0$ and $b_0 = 0$, such as the autoregressive processes with line spectrum.
2. Case of $m > n$, such as the moving average processes.

These will be covered in next few slides.

Definition (Line spectra) A line spectrum only consists of lines, i.e.,

$$S(\omega) = 2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i).$$

- The autocorrelation function of a process with line spectrum is:

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \right) e^{j\omega\tau} d\omega = \sum_i \sigma_i^2 e^{j\omega_i\tau}.$$

- An exemplified process that results in a line spectrum is:

$$\mathbf{x}(t) = \sum_i \mathbf{c}_i e^{j\omega_i t},$$

where $\{\mathbf{c}_i\}$ are uncorrelated with zero mean, and $\sigma_i^2 = E\{|\mathbf{c}_i|^2\}$.

Discrete Line Spectra

11-50

Definition (Discrete line spectra) A line spectrum for discrete processes only consists of lines, i.e.,

$$S[\omega] = 2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \text{ for } -\pi \leq \omega < \pi,$$

where each $-\pi \leq \omega_i < \pi$.

- The autocorrelation function of a discrete process with line spectrum is:

$$R[\tau] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(2\pi \sum_i \sigma_i^2 \delta(\omega - \omega_i) \right) e^{j\omega\tau} d\omega = \sum_i \sigma_i^2 e^{j\omega_i\tau}.$$

- An exemplified process that results in a line spectrum is:

$$\mathbf{x}[t] = \sum_i \mathbf{c}_i e^{j\omega_i t},$$

where $\{\mathbf{c}_i\}$ are uncorrelated with zero mean, and $\sigma_i^2 = E\{|\mathbf{c}_i|^2\}$.

Discrete Line Spectra

11-51

Example of AR processes with line spectra

- Suppose that

$$\mathbf{x}[t] = \sum_{i=1}^n \mathbf{c}_i e^{j\omega_i t},$$

where $\{\mathbf{c}_i\}$ are real and uncorrelated with zero mean and variance $\sigma_i^2 = E\{\mathbf{c}_i^2\}$, and each $-\pi \leq \omega_i < \pi$.

- Let $z_i = e^{j\omega_i}$.
- Find a_1, a_2, \dots, a_n such that

$$\begin{bmatrix} 1 & z_1^{-1} & z_1^{-2} & \cdots & z_1^{-n} \\ 1 & z_2^{-1} & z_2^{-2} & \cdots & z_2^{-n} \\ 1 & z_3^{-1} & z_3^{-2} & \cdots & z_3^{-n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_n^{-1} & z_n^{-2} & \cdots & z_n^{-n} \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then

$$\begin{aligned} & \mathbf{x}[t] + a_1 \mathbf{x}[t-1] + a_2 \mathbf{x}[t-2] + \cdots + a_n \mathbf{x}[t-n] \\ &= \sum_{i=1}^n \mathbf{c}_i z_i^t (1 + a_1 z_i^{-1} + \cdots + a_n z_i^{-n}) = 0. \end{aligned}$$

Discrete Line Spectra

11-52

Specifically, if $n = 2$, we require

$$\begin{aligned}D(z_1) &= 1 + a_1 z_1^{-1} + a_2 z_1^{-2} = 0 \\D(z_2) &= 1 + a_1 z_2^{-1} + a_2 z_2^{-2} = 0\end{aligned}$$

Then, $a_1 = -(z_1 + z_2)$ and $a_2 = z_1 z_2$.

If $n = 3$, we require

$$\begin{aligned}D(z_1) &= 1 + a_1 z_1^{-1} + a_2 z_1^{-2} + a_3 z_1^{-3} = 0 \\D(z_2) &= 1 + a_1 z_2^{-1} + a_2 z_2^{-2} + a_3 z_2^{-3} = 0 \\D(z_3) &= 1 + a_1 z_3^{-1} + a_2 z_3^{-2} + a_3 z_3^{-3} = 0\end{aligned}$$

Then, $a_1 = -(z_1 + z_2 + z_3)$, $a_2 = z_1 z_2 + z_1 z_3 + z_2 z_3$ and $a_3 = -z_1 z_2 z_3$.

In fact, $D(z) = \prod_{i=1}^n (1 - z_i z^{-1})$.

- This turns out to be a special case of the AR processes for which $b_0 = 0$ and $D(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}$. It is usually referred to as the predictable process.

Definition (Predictable process) A process is called *predictable* if its present value can be determined by its past.

Autocorrelation and line power spectrum of $x[t]$

•

$$\begin{aligned} R_{xx}[\tau] &= E \left[\left(\sum_{i=1}^n \mathbf{c}_i e^{j\omega_i(t+\tau)} \right) \left(\sum_{k=1}^n \mathbf{c}_k^* e^{-j\omega_k t} \right) \right] \\ &= \sum_{i=1}^n \sum_{k=1}^n E [\mathbf{c}_i \mathbf{c}_k^*] e^{j\omega_i(t+\tau)} e^{-j\omega_k t} \\ &= \sum_{i=1}^n \sigma_i^2 e^{j\omega_i \tau} \end{aligned}$$

and for $-\pi \leq \omega < \pi$,

$$\begin{aligned} S_{xx}[\omega] &= \sum_{\tau=-\infty}^{\infty} R_{xx}[\tau] e^{-j\omega \tau} = \sum_{\tau=-\infty}^{\infty} \sum_{i=1}^n \sigma_i^2 e^{j\omega_i \tau} e^{-j\omega \tau} \\ &= \sum_{i=1}^n \sigma_i^2 \sum_{\tau=-\infty}^{\infty} e^{-j(\omega - \omega_i)\tau} = 2\pi \sum_{i=1}^n \sigma_i^2 \delta(\omega - \omega_i). \quad (\text{Line spectra!}) \end{aligned}$$

where $\sum_{\tau=-\infty}^{\infty} e^{-j(\omega - \omega_i)\tau} = \sum_{\tau=-\infty}^{\infty} 2\pi \cdot \delta(\omega - \omega_i + 2\pi\tau)$.

Definition (MA processes) The discrete process $\mathbf{x}[t]$ is called moving average (MA) if its innovation filter is of the form:

$$L[z] = b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}.$$

Autocorrelation function of MA processes

- For an MA process,

$$\mathbf{x}[t] = b_0 \mathbf{i}[t] + b_1 \mathbf{i}[t - 1] + \cdots + b_m \mathbf{i}[t - m].$$

- Hence, the symmetric autocorrelation function (i.e., $R_{xx}[\tau] = R_{xx}[-\tau]$) equals

$$\begin{aligned} R_{xx}[\tau] &= E\{\mathbf{x}[t + \tau]\mathbf{x}[t]\} \\ &= E\left\{\left(\sum_{i=0}^m b_i \mathbf{i}[t + \tau - i]\right)\left(\sum_{k=0}^m b_k \mathbf{i}[t - k]\right)\right\} \\ &= \sum_{i=0}^m \sum_{k=0}^m b_i b_k E\{\mathbf{i}[t + \tau - i]\mathbf{i}[t - k]\} = \sum_{i=0}^m \sum_{k=0}^m b_i b_k \delta[\tau - i + k] \\ &= \begin{cases} \sum_{k=0}^{m-\tau} b_{k+\tau} b_k, & \text{for } 0 \leq \tau \leq m \\ 0, & \text{for } \tau > m \end{cases} \end{aligned}$$

Autoregressive Moving Average Processes

11-55

Definition (ARMA processes) The discrete process $\mathbf{x}[t]$ is called autoregressive moving average (ARMA) if its innovation filter is of the form:

$$\mathbf{L}[z] = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \frac{N[z]}{D[z]}.$$

- The analysis of the ARMA processes has been done; so we omit it. See the slides after Slide 11-43.

The end of Section 11-2 Finite-Order Systems and State Variables

11-3 Fourier Series and Karhunen-Loève Expansions

11-56

Question: Given that $\omega_0 = 2\pi/T$,

$$\hat{\mathbf{x}}(t) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{jn\omega_0 t} \quad \text{and} \quad \mathbf{c}_n = \frac{1}{T} \int_0^T \mathbf{x}(t) e^{-jn\omega_0 t} dt,$$

whether does $\hat{\mathbf{x}}(t)$ well approximate the WSS $\mathbf{x}(t)$?

Theorem $\hat{\mathbf{x}}(t)$ equals $\mathbf{x}(t)$ for $0 < t < T$ in the MS sense, i.e.,

$$E[|\hat{\mathbf{x}}(t) - \mathbf{x}(t)|^2] = 0$$

for $0 < t < T$.

Proof: Observe that for $0 < t < T$,

$$\begin{aligned} E[|\hat{\mathbf{x}}(t)|^2] &= E \left[\left(\sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{jn\omega_0 t} \right) \left(\sum_{m=-\infty}^{\infty} \mathbf{c}_m^* e^{-jm\omega_0 t} \right) \right] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[\mathbf{c}_n \mathbf{c}_m^*] e^{jn\omega_0 t} e^{-jm\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left(\frac{1}{T^2} \int_0^T \int_0^T E[\mathbf{x}(u) \mathbf{x}^*(v)] e^{-jn\omega_0 u} e^{jm\omega_0 v} dudv \right) e^{jn\omega_0 t} e^{-jm\omega_0 t} \end{aligned}$$

Fourier Series

11-57

(continued)

$$\begin{aligned} \left(E[|\hat{\mathbf{x}}(t)|^2] \right) &= \int_0^T \int_0^T R_{xx}(u-v) \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0(t-u)} \right) \left(\frac{1}{T} \sum_{m=-\infty}^{\infty} e^{jm\omega_0(v-t)} \right) dudv \\ &= \int_0^T \int_0^T R_{xx}(u-v) \left(\sum_{n=-\infty}^{\infty} \delta(t-u+nT) \right) \left(\sum_{m=-\infty}^{\infty} \delta(v-t+mT) \right) dudv \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_0^T \int_0^T R_{xx}(u-v) \delta(t-u+nT) \delta(v-t+mT) dudv \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \int_0^T R_{xx}(t+nT-v) \mathbf{1}\{0 < t+nT < T\} \delta(v-t+mT) dv \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_{xx}((n+m)T) \mathbf{1}\{0 < t+nT < T\} \mathbf{1}\{0 < t-mT < T\} \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} R_{xx}((n+m)T) \mathbf{1} \left\{ -\frac{t}{T} < n < 1 - \frac{t}{T} \text{ and } \frac{t}{T} - 1 < m < \frac{t}{T} \right\} \\ &= R_{xx}(0) = E[|\mathbf{x}(t)|^2], \end{aligned}$$

Fourier Series

11-58

and

$$\begin{aligned} E[\hat{\mathbf{x}}(t)\mathbf{x}^*(t)] &= E\left[\left(\sum_{n=-\infty}^{\infty} \mathbf{c}_n e^{jn\omega_0 t}\right) \mathbf{x}^*(t)\right] \\ &= \sum_{n=-\infty}^{\infty} E[\mathbf{c}_n \mathbf{x}^*(t)] e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} E\left[\left(\frac{1}{T} \int_0^T \mathbf{x}(s) e^{-jn\omega_0 s} ds\right) \mathbf{x}^*(t)\right] e^{jn\omega_0 t} \\ &= \int_0^T E[\mathbf{x}(s)\mathbf{x}^*(t)] \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0(t-s)}\right) ds \\ &= \int_0^T R_{xx}(s-t) \left(\sum_{n=-\infty}^{\infty} \delta(t-s+nT)\right) ds \\ &= \sum_{n=-\infty}^{\infty} R_{xx}(nT) \mathbf{1}\{0 < t+nT < T\} \quad (\text{i.e., } \frac{t}{T} - 1 < -n < \frac{t}{T}) \\ &= R_{xx}(0) \end{aligned}$$

Fourier Series

11-59

Similarly,

$$\begin{aligned} E[\hat{\mathbf{x}}^*(t)\mathbf{x}(t)] &= E\left[\left(\sum_{n=-\infty}^{\infty} \mathbf{c}_n^* e^{-jn\omega_0 t}\right) \mathbf{x}(t)\right] \\ &= \sum_{n=-\infty}^{\infty} E[\mathbf{c}_n^* \mathbf{x}(t)] e^{-jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} E\left[\left(\frac{1}{T} \int_0^T \mathbf{x}^*(s) e^{jn\omega_0 s} ds\right) \mathbf{x}(t)\right] e^{-jn\omega_0 t} \\ &= \int_0^T E[\mathbf{x}(t)\mathbf{x}^*(s)] \left(\frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_0(s-t)}\right) ds \\ &= \int_0^T R_{xx}(t-s) \left(\sum_{n=-\infty}^{\infty} \delta(s-t+nT)\right) ds \\ &= \sum_{n=-\infty}^{\infty} R_{xx}(nT) \mathbf{1}\{0 < t - nT < T\} \quad (\text{i.e., } \frac{t}{T} - 1 < n < \frac{t}{T}) \\ &= R_{xx}(0) \end{aligned}$$

Hence,

$$\begin{aligned} E[|\hat{\mathbf{x}}(t) - \mathbf{x}(t)|^2] &= E[|\hat{\mathbf{x}}(t)|^2] - E[\hat{\mathbf{x}}(t)\mathbf{x}^*(t)] - E[\hat{\mathbf{x}}^*(t)\mathbf{x}(t)] + E[|\mathbf{x}(t)|^2] \\ &= R_{xx}(0) - R_{xx}(0) - R_{xx}(0) + R_{xx}(0) \\ &= 0. \end{aligned}$$

□

Remarks

- It is tricky to say the theorem holds at $t = 0$ (respectively, $t = T$) since $\int_0^T \delta(s)ds$ or (respectively, $\int_0^T \delta(s - T)ds$) is actually indeterminate.
- It can be similarly proved that if $\mathbf{x}(t)$ is MS-periodic with period T ,

$$E[|\hat{\mathbf{x}}(t) - \mathbf{x}(t)|^2] = 0 \quad \text{for a.e. } t \in \mathfrak{R}.$$

Definition A process $\mathbf{x}(t)$ is called *MS periodic* if

$$E[|\mathbf{x}(t + T) - \mathbf{x}(t)|^2] = 0$$

for every t .

Theorem 9-1 A process $\mathbf{x}(t)$ is *MS periodic* if, and only if, its autocorrelation function is *doubly periodic*, namely,

$$R_{xx}(t_1 + mT, t_2 + nT) = R_{xx}(t_1, t_2) \text{ for every integer } m \text{ and } n.$$

- In addition, for a MS-periodic WSS process $\mathbf{x}(t)$,

$$\left\{ \mathbf{c}_n = \frac{1}{T} \int_0^T \mathbf{x}(t) e^{-jn\omega_0 t} dt \right\}_{n=-\infty}^{\infty}$$

are uncorrelated with zero-mean except possibly **non-zero-mean** at $n = 0$.

- These remarks are summarized into the next theorem.

Fourier Series

11-62

Theorem 11-1 For a MS-periodic (with period T) WSS process $\mathbf{x}(t)$, $\hat{\mathbf{x}}(t)$ equals $\mathbf{x}(t)$ in the MS sense, i.e., $E[|\hat{\mathbf{x}}(t) - \mathbf{x}(t)|^2] = 0$.

In addition, $\{\mathbf{c}_n\}_{n=-\infty}^{\infty}$ are uncorrelated with zero mean except possibly for $n = 0$.

Proof: It remains to prove that $\{\mathbf{c}_n\}_{n=-\infty}^{\infty}$ are uncorrelated with zero mean except possibly for $n = 0$.

For an MS-periodic WSS $\mathbf{x}(t)$,

$$E[\mathbf{c}_n] = \frac{1}{T} \int_0^T E[\mathbf{x}(t)] e^{-jn\omega_0 t} dt = \mu_x \delta[n], \quad (\text{because } \omega_0 = T/(2\pi))$$

Fourier Series

11-63

and

$$\begin{aligned} E[\mathbf{c}_n \mathbf{c}_m^*] &= E \left[\left(\frac{1}{T} \int_0^T \mathbf{x}(t) e^{-jn\omega_0 t} dt \right) \left(\frac{1}{T} \int_0^T \mathbf{x}(s) e^{-jm\omega_0 s} ds \right)^* \right] \\ &= \frac{1}{T^2} \int_0^T \int_0^T E[\mathbf{x}(t) \mathbf{x}^*(s)] e^{-jn\omega_0 t} e^{jm\omega_0 s} dt ds \\ &= \frac{1}{T^2} \int_0^T \int_0^T R_{xx}(t-s) e^{-jn\omega_0 t} e^{jm\omega_0 s} dt ds, \quad u = t-s \\ &= \frac{1}{T^2} \int_0^T \int_{-s}^{T-s} R_{xx}(u) e^{-jn\omega_0(u+s)} e^{jm\omega_0 s} du ds \\ &= \left(\frac{1}{T} \int_0^T e^{-j(n-m)\omega_0 s} ds \right) \left(\frac{1}{T} \int_0^T R_{xx}(u) e^{-jn\omega_0 u} du \right) \\ &= \delta[n-m] \left(\frac{1}{T} \int_0^T R_{xx}(u) e^{-jn\omega_0 u} du \right) \quad (\text{since } \omega_0 = T/(2\pi)) \end{aligned}$$

□

Remarks

- $\{\mathbf{c}_n\}_{n=-\infty}^{\infty}$ may not be uncorrelated if $\mathbf{x}(t)$ is not MS-periodic!
- Even if $\mathbf{x}(t)$ is MS-periodic, $\{\mathbf{c}_n\}_{n=-\infty}^{\infty}$ may not be uncorrelated when the chosen T is not the MS-period for $\mathbf{x}(t)$.
- **Concern:** Can we find an alternative expression for $\mathbf{x}(t)$ for which the coefficients are guaranteed to be **uncorrelated**?

Answer: Karhunen-Loève Expansions.

Karhunen-Loève Expansions

11-65

Question: Given a set of orthonormal functions $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ over $[0, T)$, define

$$\hat{\mathbf{x}}(t) = \sum_{n=-\infty}^{\infty} \mathbf{c}_n \varphi_n(t) \quad \text{and} \quad \mathbf{c}_n = \int_0^T \mathbf{x}(t) \varphi_n^*(t) dt.$$

Whether does $\hat{\mathbf{x}}(t)$ well approximate $\mathbf{x}(t)$?

Theorem $\{\mathbf{c}_n\}_{n=-\infty}^{\infty}$ are **orthogonal**, if

$$\int_0^T R_{xx}(t, s) \varphi_n(s) ds = \lambda_n \varphi_n(t)$$

for some λ_n for every n .

For MS-periodic WSS process $\mathbf{x}(t)$ with MS-period T ,

$$\begin{aligned} \int_0^T R_{xx}(t-s) \left(\frac{1}{\sqrt{T}} e^{jn\omega_0 s} \right) ds &= \int_{t-T}^t R_{xx}(u) \frac{1}{\sqrt{T}} e^{jn\omega_0(t-u)} du \\ &= \frac{1}{\sqrt{T}} e^{jn\omega_0 t} \int_0^T R_{xx}(u) e^{-jn\omega_0 u} du = \lambda_n \left(\frac{1}{\sqrt{T}} e^{jn\omega_0 t} \right), \end{aligned}$$

where $\omega_0 = 2\pi/T$ and $\lambda_n = \int_0^T R_{xx}(u) e^{-jn\omega_0 u} du$.

Proof:

$$\begin{aligned} E[\mathbf{c}_n \mathbf{c}_m^*] &= E \left[\left(\int_0^T \mathbf{x}(t) \varphi_n^*(t) dt \right) \left(\int_0^T \mathbf{x}(s) \varphi_m^*(s) ds \right)^* \right] \\ &= \int_0^T \int_0^T E[\mathbf{x}(t) \mathbf{x}^*(s)] \varphi_n^*(t) \varphi_m(s) dt ds \\ &= \int_0^T \left(\int_0^T R_{xx}(t, s) \varphi_m(s) ds \right) \varphi_n^*(t) dt \\ &= \int_0^T \lambda_m \varphi_m(t) \varphi_n^*(t) dt \\ &= \lambda_m \delta[m - n]. \end{aligned}$$

□

Remarks

- $\{\varphi_n(t)\}_{n=-\infty}^{\infty}$ and $\{\lambda_n\}_{n=-\infty}^{\infty}$ are respectively called the eigenfunctions and eigenvalues of $R_{xx}(t, s)$.

Karhunen-Loève Expansions

11-67

- For a random process $\mathbf{x}(t)$, projection λ_n is real and non-negative for every n .

Proof:

$$\begin{aligned} E \left[\left| \int_0^T \mathbf{x}(t) \varphi_n^*(t) dt \right|^2 \right] &= E \left[\left(\int_0^T \mathbf{x}(t) \varphi_n^*(t) dt \right) \left(\int_0^T \mathbf{x}^*(s) \varphi_n(s) ds \right) \right] \\ &= \int_0^T \left(\int_0^T R_{xx}(t, s) \varphi_n(s) ds \right) \varphi_n^*(t) dt \\ &= \int_0^T \lambda_n \varphi_n(t) \varphi_n^*(t) dt \\ &= \lambda_n. \end{aligned}$$

□

- $R_{xx}(t, t) = \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2$ for $0 \leq t < T$. (Property of the eigen-system)

Karhunen-Loève Expansions

11-68

Theorem $E[|\hat{\mathbf{x}}(t) - \mathbf{x}(t)|^2] = 0$ for $0 < t < T$.

Proof: Observe that

$$\begin{aligned} E[|\hat{\mathbf{x}}(t)|^2] &= E \left[\left(\sum_{n=-\infty}^{\infty} \mathbf{c}_n \varphi_n(t) \right) \left(\sum_{m=-\infty}^{\infty} \mathbf{c}_m^* \varphi_m^*(t) \right) \right] \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} E[\mathbf{c}_n \mathbf{c}_m^*] \varphi_n(t) \varphi_m^*(t) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \lambda_m \delta[m - n] \varphi_n(t) \varphi_m^*(t) \quad (\text{because } \{\mathbf{c}_n\} \text{ orthogonal}) \\ &= \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2 \quad (= R_{xx}(t, t)) \end{aligned}$$

Karhunen-Loève Expansions

11-69

and

$$\begin{aligned} E[\hat{\mathbf{x}}(t)\mathbf{x}^*(t)] &= E\left[\left(\sum_{n=-\infty}^{\infty} \mathbf{c}_n\varphi_n(t)\right)\mathbf{x}^*(t)\right] \\ &= \sum_{n=-\infty}^{\infty} E[\mathbf{c}_n\mathbf{x}^*(t)]\varphi_n(t) \\ &= \sum_{n=-\infty}^{\infty} E\left[\left(\int_0^T \mathbf{x}(s)\varphi_n^*(s)ds\right)\mathbf{x}^*(t)\right]\varphi_n(t) \\ &= \sum_{n=-\infty}^{\infty} \left(\int_0^T R_{xx}(t,s)\varphi_n(s)ds\right)^* \varphi_n(t) \\ &= \sum_{n=-\infty}^{\infty} \lambda_n^* \varphi_n^*(t)\varphi_n(t) \\ &= \sum_{n=-\infty}^{\infty} \lambda_n^* |\varphi_n(t)|^2 = \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2 \quad (\lambda_n \text{ real and non-negative}) \end{aligned}$$

Similarly,

$$E[\hat{\mathbf{x}}^*(t)\mathbf{x}(t)] = \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2.$$

Karhunen-Loève Expansions

11-70

Hence,

$$\begin{aligned} E[|\hat{\mathbf{x}}(t) - \mathbf{x}(t)|^2] &= E[|\hat{\mathbf{x}}(t)|^2] - E[\hat{\mathbf{x}}(t)\mathbf{x}^*(t)] - E[\hat{\mathbf{x}}^*(t)\mathbf{x}(t)] + E[|\mathbf{x}(t)|^2] \\ &= \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2 - \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2 - \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2 + R_{xx}(t, t) \\ &= R_{xx}(t, t) - \sum_{n=-\infty}^{\infty} \lambda_n |\varphi_n(t)|^2, \end{aligned}$$

which equals zero by **property of the eigen-system**. □

Mercer's Theorem tells that $R_{xx}(t, s) = \sum_{n=-\infty}^{\infty} \lambda_n \varphi_n(t) \varphi_n^*(s)$.

Example 11-10: Wiener process. Suppose

- $\mathbf{n}[0, 0) = 0$,
- $\mathbf{n}[t_1, t_2)$ is Gaussian distributed with mean zero and variance $\alpha(t_2 - t_1)$,
- and $\mathbf{n}[t_1, t_2)$ and $\mathbf{n}[t_3, t_4)$ are independent if $[t_1, t_2)$ and $[t_3, t_4)$ are non-overlapping intervals.

Please determine the Karhunen-Loève expansion of real process $\mathbf{x}(t) \triangleq \mathbf{n}[0, t)$.

Karhunen-Loève Expansions

11-71

Answer:

•

$$\begin{aligned}
 R_{xx}(t_1, t_2) &= E[\mathbf{x}(t_1)\mathbf{x}^*(t_2)] \\
 &= E[\mathbf{n}[0, t_1]\mathbf{n}[0, t_2)] \\
 &= E[(\mathbf{n}[0, t_{\min}) + \mathbf{n}[t_{\min}, t_{\max}))\mathbf{n}[0, t_{\min}]] \\
 &= E[\mathbf{n}^2[0, t_{\min}]] + E[\mathbf{n}[t_{\min}, t_{\max})\mathbf{n}[0, t_{\min}]] \\
 &= E[\mathbf{n}^2[0, t_{\min}]] + \cancel{E[\mathbf{n}[t_{\min}, t_{\max})]E[\mathbf{n}[0, t_{\min}]]} \\
 &= \alpha \min\{t_1, t_2\},
 \end{aligned}$$

where $t_{\min} \triangleq \min\{t_1, t_2\}$ and $t_{\max} \triangleq \max\{t_1, t_2\}$.

•

$$\begin{aligned}
 \int_0^T R_{xx}(t, s)\varphi(s)ds &= \lambda\varphi(t) \Leftrightarrow \alpha \int_0^T \min\{t, s\}\varphi(s)ds = \lambda\varphi(t) \\
 \Leftrightarrow \alpha \int_0^t s\varphi(s)ds + \alpha t \int_t^T \varphi(s)ds &= \lambda\varphi(t) \quad \text{(a1)} \\
 \Leftrightarrow \begin{cases} \alpha \int_t^T \varphi(s)ds = \lambda\varphi'(t) & \text{(a2)} \\ \lambda\varphi''(t) + \alpha\varphi(t) = 0 \end{cases} & \text{with initially } \begin{cases} \text{(a1)} \varphi(0) = 0 \\ \text{(a2)} \varphi'(T) = 0 \end{cases}
 \end{aligned}$$

Karhunen-Loève Expansions

11-72

Theorem 8.6 [Tom M. Apostol, *Calculus*, pp. 326, Volume 1, 2nd Edition, 1967] The solution of the equation $y''(x) + by(x) = 0$ is

$$y(x) = c_1 u_1(x) + c_2 u_2(x),$$

where c_1 and c_2 are constants determined by initial conditions, and

1. $u_1(x) = 1$ and $u_2(x) = x$ if $b = 0$;
2. $u_1(x) = e^{kx}$ and $u_2(x) = e^{-kx}$ if $b = -k^2 < 0$;
3. $u_1(x) = \cos(kx)$ and $u_2(x) = \sin(kx)$ if $b = k^2 > 0$.

- Consequently, $\varphi_n(t) = c_1 \cos(t\sqrt{\alpha/\lambda_n}) + c_2 \sin(t\sqrt{\alpha/\lambda_n})$, and the two initial conditions give that $c_1 = 0$ ($\varphi_n(0) = 0$) and $T\sqrt{\alpha/\lambda_n} = (2k_n + 1)\pi/2$ for integer k_n ($\varphi'_n(T) = 0$). Moreover,

$$\begin{aligned} \int_0^T \varphi_n^2(t) dt &= \int_0^T c_2^2 \sin^2 \left(\frac{(2k_n + 1)\pi}{2T} t \right) dt \\ &= \int_0^1 c_2^2 \sin^2 \left(\frac{(2k_n + 1)\pi}{2} u \right) T du = \frac{T}{2} c_2^2 = 1 \end{aligned}$$

gives that $c_2 = \sqrt{2/T}$.

Karhunen-Loève Expansions

11-73

- To sum up,

$$\varphi_n(t) = \sqrt{\frac{2}{T}} \sin\left(\frac{(2k_n + 1)\pi}{2T}t\right), \quad \lambda_n = \frac{4\alpha T^2}{(2k_n + 1)^2\pi^2},$$

and

$$\begin{aligned} \int_0^T \varphi_n(t)\varphi_m^*(t)dt &= \int_0^T \frac{2}{T} \sin\left(\frac{(2k_n + 1)\pi}{2T}t\right) \sin\left(\frac{(2k_m + 1)\pi}{2T}t\right) dt \\ &= \int_0^1 2 \sin\left(\frac{(2k_n + 1)\pi}{2}u\right) \sin\left(\frac{(2k_m + 1)\pi}{2}u\right) du \\ &= \int_0^1 \cos[(k_n - k_m)\pi u]du - \int_0^1 \cos[(k_n + k_m + 1)\pi u]du \\ &= \delta[k_n - k_m] - \delta[k_n + k_m + 1] \\ &= \begin{cases} 1, & k_n = k_m \text{ (equivalently } (2k_n + 1) = (2k_m + 1)) \\ -1, & (2k_n + 1) = -(2k_m + 1) \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Karhunen-Loève Expansions

11-74

So, it only requires to take those k_n 's that make $(2k_n + 1)$ strictly positive. This concludes that the Wiener process $\mathbf{x}(t)$ for $t \in [0, T)$ can be written as a sum of sine waves:

$$\mathbf{x}(t) = \sqrt{\frac{2}{T}} \sum_{n=0}^{\infty} \mathbf{c}_n \sin\left(\frac{(2n+1)\pi}{2T}t\right)$$

and

$$\mathbf{c}_n = \sqrt{\frac{2}{T}} \int_0^T \mathbf{x}(t) \sin\left(\frac{(2n+1)\pi}{2T}t\right) dt.$$

□

By assigning $\tilde{\mathbf{c}}_n = \mathbf{c}_n \sqrt{2/T}$, we can simplify the expression as:

$$\mathbf{x}(t) = \sum_{n=0}^{\infty} \tilde{\mathbf{c}}_n \sin\left(\frac{(2n+1)\pi}{2T}t\right) \quad \text{and} \quad \tilde{\mathbf{c}}_n = \frac{2}{T} \int_0^T \mathbf{x}(t) \sin\left(\frac{(2n+1)\pi}{2T}t\right) dt.$$

Karhunen-Loève Expansions

11-75

Example. Suppose $\mathbf{x}(t)$ is WSS. Then, from

$$\int_{-\infty}^{\infty} R_{xx}(t-s)\varphi_{\lambda}(s)ds = \lambda\varphi_{\lambda}(t),$$

we know that the Fourier transform $\Phi_{\lambda}(\omega)$ of eigenfunction $\varphi_{\lambda}(t)$ and eigenvalue λ should satisfy:

$$S_{xx}(\omega)\Phi_{\lambda}(\omega) = \lambda\Phi_{\lambda}(\omega).$$

This implies

$$(S_{xx}(\omega) - \lambda)\Phi_{\lambda}(\omega) = 0.$$

Suppose $S_{xx}(\omega) = \lambda$ only at $\omega = u$. (There could be other value of ω such as $\omega = v$ that also makes $S_{xx}(v) = \lambda$. We would treat this case as the eigenvalue λ has several eigenfunctions.)

Then, $\Phi_{\lambda}(\omega) = \sqrt{2\pi}\delta(\omega - u)$ is an eigenfunction corresponding to eigenvalue λ , which implies

$$\varphi_{\lambda}(t) = \frac{1}{\sqrt{2\pi}}e^{jut}.$$

Karhunen-Loève Expansions

11-76

Hence,

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} \mathbf{c}_\lambda \varphi_\lambda(t) d\lambda = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{c}_\lambda e^{j u(\lambda) t} d\lambda$$

and

$$\mathbf{c}_\lambda = \int_{-\infty}^{\infty} \mathbf{x}(t) \varphi_\lambda^*(t) dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathbf{x}(t) e^{-j u t} dt.$$

We can redenote \mathbf{c}_λ by $\frac{1}{\sqrt{2\pi}} \mathbf{X}(u)$ and yield:

$$\mathbf{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{X}(u) e^{j u t} du \quad \text{and} \quad \mathbf{X}(u) = \int_{-\infty}^{\infty} \mathbf{x}(t) e^{-j u t} dt.$$

□

- This example justifies the viewpoint that the Fourier transform of a WSS process is simply the Karhunen-Loève expansion of this random process.
- We will show later that $E[\mathbf{X}(u)\mathbf{X}^*(v)] = 2\pi S_{xx}(u)\delta(u-v)$ (resp. $E[\mathbf{c}_{\lambda_1}\mathbf{c}_{\lambda_2}^*] = \frac{1}{2\pi}E[\mathbf{X}(u)\mathbf{X}^*(v)] = S_{xx}(u)\delta(u-v)$ for distinct eigenvalues λ_1 and λ_2).
- The eigenvalue corresponding to eigenvector $\frac{1}{\sqrt{2\pi}}e^{jut}$ is $\sqrt{2\pi}S_{xx}(u)$.
(I.e., the eigenvalue corresponding to eigenvector $\frac{1}{\sqrt{2\pi}}e^{j\omega t}$ is $\sqrt{2\pi}S_{xx}(\omega)$.)
- The eigenvectors $\frac{1}{\sqrt{2\pi}}e^{j\omega_1 t}$ and $\frac{1}{\sqrt{2\pi}}e^{j\omega_2 t}$ are orthogonal to each other (namely,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{j\omega_1 t} \frac{1}{\sqrt{2\pi}}e^{-j\omega_2 t} dt = \delta(\omega_1 - \omega_2)).$$

11-4 Spectral Representation of Random Processes

11-78

- The Fourier transform of a random process $\mathbf{x}(t)$ is also a random process, defined as:

$$\mathbf{X}(u) \triangleq \int_{-\infty}^{\infty} \mathbf{x}(t)e^{-jut} dt.$$

Lemma

- Let $R_{XX}(u_1, u_2)$ and $S_{XX}(\lambda_1, \lambda_2)$ be the autocorrelation function and two-dimensional power spectrum of $\mathbf{X}(t)$, respectively.
- Let $R_{xx}(t_1, t_2)$ and $S_{xx}(f_1, f_2)$ be the autocorrelation function and two-dimensional power spectrum of $\mathbf{x}(t)$, respectively.

Then,

$$R_{XX}(u_1, u_2) = S_{xx}(u_1, -u_2) \quad \text{and} \quad S_{XX}(\lambda_1, \lambda_2) = 4\pi^2 R_{xx}(-\lambda_1, \lambda_2).$$

Proof:

$$\begin{aligned}
 R_{XX}(u_1, u_2) &= E[\mathbf{X}(u_1)\mathbf{X}^*(u_2)] \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\mathbf{x}(t_1)\mathbf{x}^*(t_2)]e^{-j(u_1t_1-u_2t_2)} dt_1 dt_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(t_1, t_2)e^{-j[u_1t_1+(-u_2)t_2]} dt_1 dt_2 \\
 &= S_{xx}(u_1, -u_2)
 \end{aligned}$$

and

$$\begin{aligned}
 S_{XX}(\lambda_1, \lambda_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{XX}(u_1, u_2)e^{-j(\lambda_1u_1+\lambda_2u_2)} du_1 du_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xx}(u_1, -u_2)e^{-j(\lambda_1u_1+\lambda_2u_2)} du_1 du_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xx}(u_1, u'_2)e^{j[(-\lambda_1)u_1+\lambda_2u'_2]} du_1 du'_2 \\
 &= 4\pi^2 R_{xx}(-\lambda_1, \lambda_2).
 \end{aligned}$$

□

Example (Theorem 11-2: Nonstationary white noise) If

$$R_{xx}(t_1, t_2) = q(t_1)\delta(t_1 - t_2) \text{ with } q(t_1) > 0,$$

(which defines the so-called nonstationary white noise) then

$$\begin{aligned} S_{xx}(f_1, f_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(t_1, t_2) e^{-j(f_1 t_1 + f_2 t_2)} dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} q(t_1) \delta(t_1 - t_2) e^{-j(f_1 t_1 + f_2 t_2)} dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} q(t_2) e^{-j(f_1 + f_2)t_2} dt_2 \\ &= Q(f_1 + f_2) \end{aligned}$$

$$R_{XX}(u_1, u_2) = S_{xx}(u_1, -u_2) = Q(u_1 - u_2),$$

and

$$S_{XX}(\lambda_1, \lambda_2) = 4\pi^2 R_{xx}(-\lambda_1, \lambda_2) = 4\pi^2 q(-\lambda_1) \delta(-\lambda_1 - \lambda_2) = 4\pi^2 q(\lambda_2) \delta(\lambda_1 + \lambda_2).$$

From the above derivation, it is apparent that **if a nonstationary white noise $\mathbf{x}(t)$ has zero mean, then $\mathbf{X}(u)$ becomes WSS.** \square

Example If $\mathbf{x}(t)$ is WSS, then

$$\begin{aligned} S_{xx}(f_1, f_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(t_1 - t_2) e^{-j(f_1 t_1 + f_2 t_2)} dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xx}(s) e^{-j(f_1 s + f_1 t_2 + f_2 t_2)} ds dt_2 \\ &= S_{xx}(f_1) \int_{-\infty}^{\infty} e^{-j(f_1 + f_2)t_2} dt_2 \\ &= 2\pi S_{xx}(f_1) \delta(f_1 + f_2). \end{aligned}$$

Hence,

$$R_{XX}(u, v) = S_{xx}(u, -v) = 2\pi S_{xx}(u) \delta(u - v) \quad \left(\text{where } S_{xx}(u) \geq 0 \right).$$

□

In summary:

- The Fourier transform of a zero-mean nonstationary white process becomes WSS.
- The Fourier transform of a WSS process becomes nonstationary white.

Spectral Representation of Random Processes

11-82

Example If $\mathbf{x}(t)$ is real and WSS, then

$$R_{XX}(u, v) = E[\mathbf{X}(u)\mathbf{X}^*(v)] = S_{xx}(u, -v) = 2\pi S_{xx}(u)\delta(u - v).$$

Taking $u = \omega$ and $v = -\omega$ for $\omega \neq 0$, together with the fact that $\mathbf{X}(\omega) = \mathbf{X}^*(-\omega)$, yields:

$$\begin{aligned} R_{XX}(\omega, -\omega) &= E[\mathbf{X}(\omega)\mathbf{X}^*(-\omega)] \\ &= E[\mathbf{X}^2(\omega)] \\ &= E[\operatorname{Re}\{\mathbf{X}(\omega)\}^2] - E[\operatorname{Im}\{\mathbf{X}(\omega)\}^2] + 2jE[\operatorname{Re}\{\mathbf{X}(\omega)\} \cdot \operatorname{Im}\{\mathbf{X}(\omega)\}] \\ &\left(= 2\pi S_{xx}(\omega)\delta(2\omega) \right) = 0. \end{aligned}$$

This concludes:

$$E[\operatorname{Re}\{\mathbf{X}(\omega)\}^2] = E[\operatorname{Im}\{\mathbf{X}(\omega)\}^2] \quad \text{and} \quad E[\operatorname{Re}\{\mathbf{X}(\omega)\} \cdot \operatorname{Im}\{\mathbf{X}(\omega)\}] = 0.$$

□

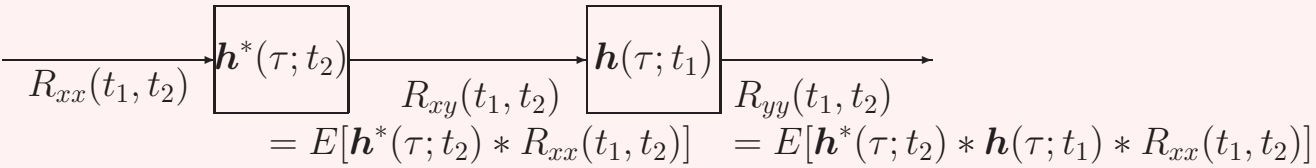
Windowing

A windowing filter is of the form $\mathbf{h}(\tau; t) = w(t)\delta(\tau)$ that induces

$$\mathbf{y}(t) = \mathbf{x}(t)w(t) = \int_{-\infty}^{\infty} \underbrace{w(t)\delta(\tau)}_{\mathbf{h}(\tau; t)} \mathbf{x}(t - \tau) d\tau$$

Example 11-11 $w(t) = \mathbf{1}\{|t| \leq T\}$ for WSS $\mathbf{x}(t)$.

Fundamental Theorem and Theorem 9-2 For any linear system,



$R_{xx}(t_1, t_2) \rightarrow \boxed{\mathbf{h}^*(\tau; t_2)} \xrightarrow{R_{xy}(t_1, t_2)} \boxed{\mathbf{h}(\tau; t_1)} \rightarrow R_{yy}(t_1, t_2)$
 $= E[\mathbf{h}^*(\tau; t_2) * R_{xx}(t_1, t_2)] = E[\mathbf{h}^*(\tau; t_2) * \mathbf{h}(\tau; t_1) * R_{xx}(t_1, t_2)]$

- For the windowing filter,

$$\begin{aligned} R_{xy}(t_1, t_2) &= E[\mathbf{h}^*(\tau; t_2) * R_{xx}(t_1, t_2)] \\ &= E \left[\int_{-\infty}^{\infty} \mathbf{h}^*(\tau; t_2) R_{xx}(t_1, t_2 - \tau) d\tau \right] \\ &= w^*(t_2) \int_{-\infty}^{\infty} \delta(\tau) R_{xx}(t_1, t_2 - \tau) d\tau \\ &= w^*(t_2) R_{xx}(t_1, t_2) \end{aligned}$$

Windowing

11-84

and

$$\begin{aligned} R_{yy}(t_1, t_2) &= E[\mathbf{h}(\tau; t_1) * R_{xy}(t_1, t_2)] \\ &= E \left[\int_{-\infty}^{\infty} \mathbf{h}(\tau; t_1) R_{xy}(t_1 - \tau, t_2) d\tau \right] \\ &= w(t_1) \int_{-\infty}^{\infty} \delta(\tau) R_{xy}(t_1 - \tau, t_2) d\tau \\ &= w(t_1) R_{xy}(t_1, t_2) \\ &\left(= w(t_1) w^*(t_2) R_{xx}(t_1, t_2) \right) \end{aligned}$$

Windowing

- For the windowing filter,

$$\begin{aligned} S_{xy}(u_1, u_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{xy}(t_1, t_2) e^{-j(t_1 u_1 + t_2 u_2)} dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_2) R_{xx}(t_1, t_2) e^{-j(t_1 u_1 + t_2 u_2)} dt_1 dt_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w^*(t_2) \left(\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xx}(v_1, v_2) e^{j(t_1 v_1 + t_2 v_2)} dv_1 dv_2 \right) e^{-j(t_1 u_1 + t_2 u_2)} dt_1 dt_2 \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-jt_1(u_1 - v_1)} dt_1 \left(\int_{-\infty}^{\infty} w(t_2) e^{-jt_2(v_2 - u_2)} dt_2 \right)^* \right) S_{xx}(v_1, v_2) dv_1 dv_2 \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2\pi \delta(u_1 - v_1) W^*(v_2 - u_2) \right) S_{xx}(v_1, v_2) dv_1 dv_2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W^*(v_2 - u_2) S_{xx}(u_1, v_2) dv_2, \end{aligned}$$

Windowing

11-86

and

$$\begin{aligned}
 S_{yy}(u_1, u_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{yy}(t_1, t_2) e^{-j(t_1 u_1 + t_2 u_2)} dt_1 dt_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(t_1) R_{xy}(t_1, t_2) e^{-j(t_1 u_1 + t_2 u_2)} dt_1 dt_2 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w(t_1) \left(\frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S_{xy}(v_1, v_2) e^{j(t_1 v_1 + t_2 v_2)} dv_1 dv_2 \right) e^{-j(t_1 u_1 + t_2 u_2)} dt_1 dt_2 \\
 &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} w(t_1) e^{-jt_1(u_1 - v_1)} dt_1 \int_{-\infty}^{\infty} e^{-jt_2(u_2 - v_2)} dt_2 \right) S_{xy}(v_1, v_2) dv_1 dv_2 \\
 &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(W(u_1 - v_1) 2\pi \delta(u_2 - v_2) \right) S_{xy}(v_1, v_2) dv_1 dv_2 \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W(u_1 - v_1) S_{xy}(v_1, u_2) dv_1 \\
 &\left(= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(u_1 - v_1) W^*(v_2 - u_2) S_{xx}(v_1, v_2) dv_1 dv_2 \right).
 \end{aligned}$$

Windowing

11-87

Hence,

$$\begin{aligned} R_{YY}(u_1, u_2) &= S_{yy}(u_1, -u_2) \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(u_1 - v_1) W^*(v_2 + u_2) S_{xx}(v_1, v_2) dv_1 dv_2. \end{aligned}$$

For WSS $\mathbf{x}(t)$, $S_{xx}(v_1, v_2) = 2\pi S_{xx}(v_1) \delta(v_1 + v_2)$ (cf. Slide 11-81); this reduces the formula of $R_{YY}(u_1, u_2)$ to:

$$\begin{aligned} R_{YY}(u_1, u_2) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(u_1 - v_1) W^*(v_2 + u_2) 2\pi S_{xx}(v_1) \delta(v_1 + v_2) dv_1 dv_2 \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W(u_1 - v_1) W^*(u_2 - v_1) S_{xx}(v_1) dv_1. \end{aligned}$$

Windowing

11-88

Example 11-11 $w(t) = \mathbf{1}\{|t| \leq T\}$ for WSS $\mathbf{x}(t)$. Determine $R_{YY}(u, u)$.

Answer: We know that $W(\omega) = 2 \sin(T\omega)/\omega$.

Hence,

$$\begin{aligned} R_{YY}(u, u) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W(u-v)W^*(u-v)S_{xx}(v)dv \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |W(u-v)|^2 S_{xx}(v)dv \\ &= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2(T(u-v))}{(u-v)^2} S_{xx}(v)dv. \end{aligned}$$

□

Fourier-Stieltjes Representation of WSS processes

11-89

Define

$$\mathbf{Z}(\omega) \triangleq \int_{-\infty}^{\omega} \mathbf{X}(\alpha) d\alpha$$

where $\mathbf{X}(\omega)$ is the Fourier transform of a WSS process $\mathbf{x}(t)$.

- By the Fourier-Stieltjes notation,

$$d\mathbf{Z}(\omega) = \mathbf{X}(\omega) d\omega.$$

Hence,

$$\mathbf{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \mathbf{X}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\mathbf{Z}(\omega).$$

Properties of $\mathbf{Z}(\omega)$

11-90

- That $\mathbf{x}(t)$ is WSS implies

$$R_{XX}(u, v) = 2\pi S_{xx}(u)\delta(u - v),$$

where $S_{xx}(u) \geq 0$, namely, $\mathbf{X}(u)$ is a nonstationary white process (cf. Slide 11-81).

- Integration of a nonstationary white process is a process with **orthogonal increments**.

Proof:

$$\begin{aligned} & E\{[\mathbf{Z}(\omega_2) - \mathbf{Z}(\omega_1)][\mathbf{Z}(\omega_4) - \mathbf{Z}(\omega_3)]^*\} \\ &= E\left\{\int_{\omega_1}^{\omega_2} \mathbf{X}(\alpha)d\alpha \cdot \int_{\omega_3}^{\omega_4} \mathbf{X}^*(\beta)d\beta\right\} \\ &= \int_{\omega_1}^{\omega_2} \int_{\omega_3}^{\omega_4} R_{XX}(\alpha, \beta)d\beta d\alpha \\ &= \int_{\omega_1}^{\omega_2} \int_{\omega_3}^{\omega_4} 2\pi S_{xx}(\alpha)\delta(\alpha - \beta)d\beta d\alpha \\ &= \int_{\omega_1}^{\omega_2} 2\pi S_{xx}(\alpha)\mathbf{1}\{\omega_3 < \alpha < \omega_4\}d\alpha \\ &= 0, \text{ if } (\omega_1, \omega_2) \cap (\omega_3, \omega_4) = \emptyset. \end{aligned}$$

Wold's Decomposition

11-91

Theorem (Wold's decomposition for continuous processes) An arbitrary WSS process $\mathbf{x}(t)$ can be decomposed into sum of a *regular* process $\mathbf{x}_r(t)$ and a *predictable* process $\mathbf{x}_p(t)$, for which $\mathbf{x}_r(t)$ and $\mathbf{x}_p(t)$ are orthogonal.

Definition (Predictable process) A process is called *predictable* if its present value can be determined by its past.

- A (WSS) process is predictable if, and only if, **its spectrum consists of lines**.
- An example of a predictable process is the discrete AR process with line spectra. See Slide 11-51:

$$\mathbf{x}[t] + a_1\mathbf{x}[t - 1] + a_2\mathbf{x}[t - 2] + \cdots + a_n\mathbf{x}[t - n] = 0.$$

Theorem (Wold's decomposition for discrete processes) An arbitrary WSS process $\mathbf{x}[t]$ can be decomposed into sum of a *regular* process $\mathbf{x}_r[t]$ and a *predictable* process $\mathbf{x}_p[t]$, for which $\mathbf{x}_r[t]$ and $\mathbf{x}_p[t]$ are orthogonal.

Wold's Decomposition

11-92

Proof:

- Form the predictor of $\mathbf{x}[t]$ based on its past as:

$$\hat{\mathbf{x}}[t] = \sum_{k=1}^{\infty} a_k \mathbf{x}[t - k].$$

The optimal $\{a_k\}_{k=1}^{\infty}$ in the MS sense can be obtained through the fact that the MS prediction error

$$\mathbf{e}[t] = \mathbf{x}[t] - \hat{\mathbf{x}}[t]$$

is orthogonal to the data, i.e.,

$$\begin{aligned} E\{\mathbf{e}[t]\mathbf{x}^*[t - m]\} &= E\left\{\left(\mathbf{x}[t] - \sum_{k=1}^{\infty} a_k \mathbf{x}[t - k]\right) \mathbf{x}^*[t - m]\right\} \\ &= 0 \text{ for any } m \geq 1. \end{aligned}$$

This leads to the discrete Wiener-Höpe equation:

$$R_{xx}[m] = \sum_{k=1}^{\infty} a_k R_{xx}[m - k] \text{ for } m > 0.$$

Wold's Decomposition

11-93

In addition, it can be shown that $\mathbf{e}[t]$ is a white process.

For $\tau > 0$,

$$E \{ \mathbf{e}[t + \tau] \mathbf{e}^*[t] \} = \underbrace{E \{ \mathbf{e}[t + \tau] \mathbf{x}^*[t] \}}_{=0} - \sum_{m=1}^{\infty} a_m \underbrace{E \{ \mathbf{e}[t + \tau] \mathbf{x}^*[t - m] \}}_{=0} = 0.$$

For $\tau < 0$,

$$E \{ \mathbf{e}[t + \tau] \mathbf{e}^*[t] \} = (E \{ \mathbf{e}[t] \mathbf{e}^*[t + \tau] \})^* = 0.$$

Hence, $\mathbf{e}[t]$ is white.

In summary,

$\hat{\mathbf{x}}[t]$ is the best MS estimate of $\mathbf{x}[t]$ in terms of the past of $\mathbf{x}[t]$.

$\mathbf{e}[t] = \mathbf{x}[t] - \hat{\mathbf{x}}[t]$ is the part of $\mathbf{x}[t]$ that remains “unestimated.”

Wold's Decomposition

11-94

- Form the best MS estimator of $\mathbf{x}[t]$ in terms of $\mathbf{e}[t]$ and its past:

$$\mathbf{x}_r[t] = \sum_{k=0}^{\infty} w_k \mathbf{e}[t - k].$$

Again, the error

$$\mathbf{x}_p[t] = \mathbf{x}[t] - \mathbf{x}_r[t] = \mathbf{x}[t] - \sum_{k=0}^{\infty} w_k \underbrace{\left(\mathbf{x}[t - k] - \sum_{\ell=1}^{\infty} \mathbf{x}[t - k - \ell] \right)}_{\mathbf{e}[t - k]}$$

should be orthogonal to $\{\mathbf{e}[t - k]\}_{k=0}^{\infty}$. Since $\mathbf{x}_p[t]$ is a linear combination of $\mathbf{x}[t]$ and its past, $\mathbf{x}_p[t]$ is orthogonal to $\mathbf{e}[t + m]$ for $m > 0$.

In summary,

$$\begin{cases} \mathbf{x}_p[t] \perp \mathbf{e}[t - k] \text{ for every integer } k \\ \mathbf{x}_r[t] = \sum_{k=0}^{\infty} w_k \mathbf{e}[t - k] \end{cases}$$

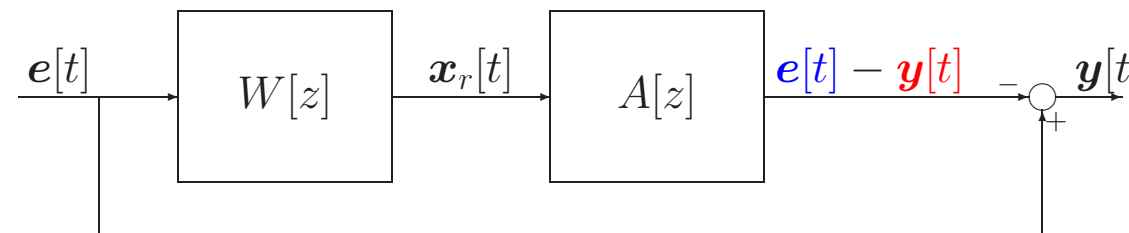
implies $\mathbf{x}_p[t] \perp \mathbf{x}_r[t]$.

- $\mathbf{x}_r[t]$ is obtained by feeding a white input to a causal (and stable) filter; hence, it is **regular**.

Wold's Decomposition

11-95

- It remains to prove that $\mathbf{x}_p[t]$ is predictable.
 - Define two filters $A[z] = 1 - \sum_{k=1}^{\infty} a_k z^{-k}$ and $W[z] = \sum_{k=0}^{\infty} w_k z^{-k}$.
 - Define $\mathbf{y}[t] = \mathbf{x}_p[t] - \sum_{k=1}^{\infty} a_k \mathbf{x}_p[t - k]$.
 - Then, by that $\mathbf{e}[t]$ and $\mathbf{y}[t]$ are respectively the outputs due to inputs $\mathbf{x}[t]$ and $\mathbf{x}_p[t]$ through linear filter $A[z]$, we learn that $\mathbf{e}[t] - \mathbf{y}[t]$ is the output due to input $\mathbf{x}_r[t] = \mathbf{x}[t] - \mathbf{x}_p[t]$ through filter $A[z]$. Together with that $\mathbf{x}_r[t]$ is the output due to input $\mathbf{e}[t]$ through filter $W[z]$, we have:



- This summarizes to that $\mathbf{y}[t]$ is the output due to input $\mathbf{e}[t]$ through filter $1 - A[z]W[z]$. So, $\mathbf{y}[t]$ is completely determined by $\mathbf{e}[t]$ and its past.
- However, the definition of $\mathbf{y}[t] = \mathbf{x}_p[t] - \sum_{k=1}^{\infty} a_k \mathbf{x}_p[t - k]$ indicates that $\mathbf{y}[t]$ is also completely determined by $\mathbf{x}_p[t]$ and its past.
- Finally, $\mathbf{x}_p[t] \perp \mathbf{e}[t - k]$ for every integer k implies $E\{|\mathbf{y}[t]|^2\} = 0$. \square

Wold's Decomposition

11-96

Further observation on Wold's Decomposition:

- $S_{xx}[e^{j\omega}] = S_{x_r x_r}[e^{j\omega}] + S_{x_p x_p}[e^{j\omega}]$, where $S_{x_r x_r}[e^{j\omega}] = |\mathbf{L}[e^{j\omega}]|^2$ for some $\mathbf{L}[e^{j\omega}]$, and $S_p[e^{j\omega}]$ is a line spectrum.

Wold's Decomposition

11-97

Example 11-12 $\mathbf{y}(t) = \mathbf{a} \cdot \mathbf{x}(t)$ with $E[\mathbf{a}] = 0$ and WSS regular $\mathbf{x}(t)$ is independent of \mathbf{a} . Find Wold's decomposition $\mathbf{y}_r(t)$ and $\mathbf{y}_p(t)$ of $\mathbf{y}(t)$.

Answer:

$$\begin{aligned} R_{yy}(\tau) &= E[\mathbf{y}(t + \tau)\mathbf{y}^*(t)] \\ &= E[\mathbf{a}\mathbf{x}(t + \tau)\mathbf{a}^*\mathbf{x}^*(t)] \\ &= \sigma_a^2 R_{xx}(\tau), \end{aligned}$$

where $\sigma_a^2 = E[\mathbf{a}\mathbf{a}^*]$. Hence,

$$S_{yy}(\omega) = \sigma_a^2 S_{xx}(\omega) = \sigma_a^2 [S_{xx}^c(\omega) + 2\pi|\eta_x|^2\delta(\omega)],$$

where $\eta_x \triangleq E[\mathbf{x}(t)]$. Accordingly,

$$S_{yy,r}(\omega) = \sigma_a^2 S_{xx}^c(\omega) \quad \text{and} \quad S_{yy,p}(\omega) = 2\pi|\eta_x|^2\sigma_a^2\delta(\omega).$$

We can then set $\mathbf{y}_p(t) = \eta_x\mathbf{a}$, and $\mathbf{y}_r(t) = \mathbf{y}(t) - \eta_x\mathbf{a} = \mathbf{a}[\mathbf{x}(t) - \eta_x]$. □

Examination of the selected $\mathbf{y}_p(t)$ and $\mathbf{y}_r(t)$:

- $\mathbf{y}_p(t) = \mathbf{y}_p(t - \tau)$ for any $\tau \geq 0$; hence, $\mathbf{y}_p(t)$ can be determined by its past.
- $E[\mathbf{y}_r(t + \tau)\mathbf{y}_r^*(t)] = \sigma_a^2 R_{xx}^c(\tau)$, and hence $S_{y_r y_r}(\omega) = \sigma_a^2 S_{xx}^c(\omega)$.
- $E[\mathbf{y}_r(t)\mathbf{y}_p^*(t)] = E\{\mathbf{a}[\mathbf{x}(t) - \eta_x]\eta_x^*\mathbf{a}^*\} = \sigma_a^2 \eta_x^* E\{\mathbf{x}(t) - \eta_x\} = 0$.

Spectral Representation of Discrete Random Processes₁₁₋₉₈

- The Fourier transform of a discrete random process $\mathbf{x}[t]$ is also a random process defined as:

$$\mathbf{X}(u) \triangleq \sum_{t=-\infty}^{\infty} \mathbf{x}[t]e^{-jut},$$

which is periodic with period 2π .

Lemma

- Let $R_{XX}(u_1, u_2)$ be the autocorrelation function of $\mathbf{X}(t)$.
- Let $S_{xx}[f_1, f_2]$ be the two-dimensional power spectrum of discrete $\mathbf{x}[t]$.

Then,

$$R_{XX}(u_1, u_2) = S_{xx}[u_1, -u_2] \text{ for } -\pi \leq u_1, u_2 < \pi.$$

Spectral Representation of Discrete Random Processes₁₁₋₉₉

Proof:

$$\begin{aligned}R_{XX}(u_1, u_2) &= E[\mathbf{X}(u_1)\mathbf{X}^*(u_2)] \\&= \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} E\{\mathbf{x}[t_1]\mathbf{x}^*[t_2]\}e^{-j(u_1t_1-u_2t_2)} \\&= \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} R_{xx}[t_1, t_2]e^{-j[u_1t_1+(-u_2)t_2]} \\&= S_{xx}[u_1, -u_2].\end{aligned}$$

□

Spectral Representation of Discrete Random Processes₁₁₋₁₀₀

Example If $\mathbf{x}[t]$ is WSS, then for $-\pi \leq f_1, f_2 < \pi$,

$$\begin{aligned}
 S_{xx}[f_1, f_2] &= \sum_{t_1=-\infty}^{\infty} \sum_{t_2=-\infty}^{\infty} R_{xx}[t_1 - t_2] e^{-j(f_1 t_1 + f_2 t_2)} \\
 &= \sum_{t_2=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} R_{xx}[s] e^{-j(f_1 s + f_1 t_2 + f_2 t_2)} \\
 &= S_{xx}[f_1] \sum_{t_2=-\infty}^{\infty} e^{-j(f_1 + f_2)t_2} \\
 &= 2\pi S_{xx}[f_1] \delta(f_1 + f_2).
 \end{aligned}$$

$$2\pi \sum_{n=-\infty}^{\infty} \delta(x + 2\pi n) = \sum_{n=-\infty}^{\infty} e^{-jnx}$$

Hence, for $-\pi \leq u, v < \pi$,

$$R_{XX}(u, v) = S_{xx}[u, -v] = 2\pi S_{xx}[u] \delta(u - v) \quad \left(\text{where } S_{xx}[u] \geq 0 \right).$$

□

Bispectra and Third Order Moments

11-101

Definition (Bispectrum) The bispectrum $\bar{S}_{xxx}(\omega_1, \omega_2)$ of a random process $\mathbf{x}(t)$ is the two-dimensional Fourier transform of its third order moment $\bar{R}_{xxx}(u, v) = R_{xxx}(t + u, t + v, t) \triangleq E[\mathbf{x}(t + u)\mathbf{x}(t + v)\mathbf{x}^*(t)]$ in u and v , where $R_{xxx}(t + u, t + v, t)$ is independent of t .

Remarks

- A case that $R_{xxx}(t + u, t + v, t)$ is independent of t is that $\mathbf{x}(t)$ is SSS (in which $R_{xxx}(t + u, t + v, t)$ only depends on the two differences).
- When only the individual statistics of system input and system output are known, their power spectrums can only be used to determine the *system amplitude* (of $H(\omega)$)!

$$S_{yy}(\omega) = |H(\omega)|^2 S_{xx}(\omega).$$

- In light of the third-order moments, the *system phase* can be identified.

$$\bar{S}_{yyy}(\omega_1, \omega_2) = \bar{S}_{xxx}(\omega_1, \omega_2) H(\omega_1) H(\omega_2) H^*(\omega_1 + \omega_2).$$

Bispectra and Third Order Moments

11-102

$$\begin{aligned}
 \bar{S}_{yyy}(\omega_1, \omega_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{R}_{yyy}(u, v) e^{-j(u\omega_1 + v\omega_2)} du dv \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\mathbf{y}(t+u)\mathbf{y}(t+v)\mathbf{y}^*(t)] e^{-j(u\omega_1 + v\omega_2)} du dv \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E \left[\left(\int_{-\infty}^{\infty} h(\tau_1)\mathbf{x}(t+u-\tau_1)d\tau_1 \right) \right. \\
 &\quad \left. \left(\int_{-\infty}^{\infty} h(\tau_2)\mathbf{x}(t+v-\tau_2)d\tau_2 \right) \left(\int_{-\infty}^{\infty} h^*(\tau_3)\mathbf{x}^*(t-\tau_3)d\tau_3 \right) \right] \\
 &\quad e^{-j(u\omega_1 + v\omega_2)} du dv \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)h^*(\tau_3)\bar{R}_{xxx}(u-\tau_1+\tau_3, v-\tau_2+\tau_3) \\
 &\quad e^{-j(u\omega_1 + v\omega_2)} du dv d\tau_1 d\tau_2 d\tau_3 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)h^*(\tau_3)\bar{R}_{xxx}(u', v') \\
 &\quad e^{-j(u'\omega_1 + \tau_1\omega_1 - \tau_3\omega_1 + v'\omega_2 + \tau_2\omega_2 - \tau_3\omega_2)} du' dv' d\tau_1 d\tau_2 d\tau_3 \\
 &= \bar{S}_{xxx}(\omega_1, \omega_2) H(\omega_1) H(\omega_2) H^*(\omega_1 + \omega_2).
 \end{aligned}$$

Bispectra and Third Order Moments

11-103

Example If $\mathbf{x}(t)$ is a SSS white process, where “white” implies $\bar{R}_{xxx}(u, v) = Q\delta(u)\delta(v)$ and $\bar{S}_{xxx}(\omega_1, \omega_2) = Q$, then

$$\bar{S}_{yyy}(\omega_1, \omega_2) = Q \cdot H(\omega_1)H(\omega_2)H^*(\omega_1 + \omega_2),$$

which implies

$$\begin{aligned}\theta(\omega_1, \omega_2) &\triangleq \angle \bar{S}_{yyy}(\omega_1, \omega_2) = \angle H(\omega_1) + \angle H(\omega_2) - \angle H(\omega_1 + \omega_2) \\ &\triangleq \varphi(\omega_1) + \varphi(\omega_2) - \varphi(\omega_1 + \omega_2).\end{aligned}$$

Then

$$\left. \frac{\partial \theta(\omega_1, \omega_2)}{\partial \omega_2} \right|_{\omega_2=0} = \varphi'(0) - \varphi'(\omega_1),$$

and

$$\begin{aligned}\varphi(\omega) - \varphi(0) &= \int_0^\omega \varphi'(\omega_1) d\omega_1 \\ &= \varphi'(0)\omega - \int_0^\omega \left. \frac{\partial \theta(\omega_1, \omega_2)}{\partial \omega_2} \right|_{\omega_2=0} d\omega_1.\end{aligned}$$

Note that for a real system, $\varphi(0) = 0$. However, $\varphi'(0)$ may not be zero!

Theorem 11-4 For a real SSS process $\mathbf{x}(t)$,

$$R_{XXX}(u, v, \omega) = E[\mathbf{X}(u)\mathbf{X}(v)\mathbf{X}^*(\omega)] = 2\pi\bar{S}_{xxx}(u, v)\delta(u + v - \omega).$$

Proof:

$$\begin{aligned} E[\mathbf{X}(u)\mathbf{X}(v)\mathbf{X}^*(\omega)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E[\mathbf{x}(t_1)\mathbf{x}(t_2)\mathbf{x}^*(t_3)]e^{-j(ut_1+vt_2-\omega t_3)} dt_1 dt_2 dt_3 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{R}_{xxx}(t_1 - t_3, t_2 - t_3)e^{-j(ut_1+vt_2-\omega t_3)} dt_1 dt_2 dt_3 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{R}_{xxx}(s_1, s_2)e^{-j(us_1+vs_2+\omega t_3-\omega t_3)} ds_1 ds_2 dt_3 \\ &= 2\pi\bar{S}_{xxx}(u, v)\delta(u + v - \omega). \end{aligned}$$

□

The end of Section 11-4 Spectral Representation of Random Processes